Intermediate Finance

Lecture Slides

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Overview of Topics

Intermediate Finance
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Lecture 01: Math Refresher (Optional Self-Study)
Overview of Topics

1.1. Mathematical Prerequisites
1.2. Pre-Calculus Refresher
1.3. Calculus Refresher
1.4. Statistics Refresher

Reading: Stitz and Zeager (2013, optional, Ch 1, 6, & 9), Hillier et al. (2016, optional, Ch 4 & 5), Hartman et al. (2018, optional, Ch 1 & 2), Diez et al. (2015, July, optional, Ch 2 & 7)

An optional math refresher

Welcome to Intermediate Finance! We have an exciting term planned for you, and we offer this optional math refresher to set you up for success.

This material should help you decide if you meet the prerequisites for the module, or if you will need to refresh selective topics at the start of term.

Feel no obligation to solve every problem in this refresher. Instead, revisit the refresher whenever you encounter unfamiliar math in future lectures.

The mathematical prerequisites

Pre-Calculus: basic operations ($+, -, \times, \div$); polynomial, logarithmic, and exponential functions; summation notation; geometric series

Calculus: differentiating and finding stationary points of simple functions, such as power, exponential, and logarithmic functions; product and chain rules

Statistics: expected value, standard deviation and variance, correlation and covariance; normal distribution; linear regression

Question 1

How comfortable do you feel with the mathematical prerequisites for this module, which are listed on the previous slide?

A. I'm very comfortable with the math prerequisites, I need little to no refresher

B. I'm somewhat comfortable, but I'll need to refresh a few selective topics

C. I'm uncomfortable with the math prerequisites, I need to refresh most topics

Lecture 01: Math Refresher (Optional Self-Study)
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Open-Source Pre-Calculus Textbook: Stitz and Zeager (2013)

Our main reference for refreshing pre-calculus is the free, open-source textbook by Stitz and Zeager (2013), available to download here.

See the textbook for a review of the following topics:

- Functions: Chapters 1.3-1.4
- Exponential and Logarithmic Functions: Chapters 6.1-6.2
- Summation Notation: Chapters 9.1-9.2

This refresher material is entirely optional, we hope you find it useful!

Stitz and Zeager (2013): Some useful exercises

- Functions: 1.3.1 Exercises 1–53 and 1.4.2 Exercises 1–76
- Exponentials and Logarithms: 6.1.1 Exercises 1–77 and 6.2.1 Exercises 1–45
- Summation Notation: 9.2.1 Exercises 1–36

Remarks:

- To test your ability, try solving a few problems from each section listed above
- Solutions to all of the above questions are available in the textbook

Lecture 01: Math Refresher (Optional Self-Study)
Overview of Topics

Reading: Stitz and Zeager (2013, optional, Ch 1, 6, & 9), Hillier et al. (2016, optional, Ch 4 & 5), Hartman et al. (2018, optional, Ch 1 & 2), Diez et al. (2015, July, optional, Ch 2 & 7)

Open-Source Calculus Textbook: Hartman et al. (2018)

Our main reference for refreshing calculus is the free, open-source textbook by Hartman et al. (2018), available to download here.

See the textbook for a review of the following topics:

- Limits: Chapters 1.1-1.3
- Derivatives: Chapters 2.1-2.6

The refresher material is entirely optional, we hope you find it useful!

Hartman et al. (2018): Some useful exercises

- Limits: Exercises 1.3 (odds have solutions)
- Basic Rules of Differentiation: Exercises 2.3 (odds have solutions)
- Quotient Rule: Exercises 2.4 (odds have solutions)
- Chain Rule: Exercises 2.5 (odds have solutions)

Remarks:

- To test your ability, try solving a few problems from each section listed above
- Solutions to odd-numbered questions are available in the textbook

Lecture 01: Math Refresher (Optional Self-Study)
Overview of Topics

Reading: Stitz and Zeager (2013, optional, Ch 1, 6, & 9), Hillier et al. (2016, optional, Ch 4 & 5), Hartman et al. (2018, optional, Ch 1 & 2), Diez et al. (2015, July, optional, Ch 2 & 7)

Open-Source Statistics Textbook: Diez et al. (2015, July)

Our main reference for refreshing statistics is the free, open-source textbook by Diez et al. (2015, July), available to download here.

See the textbook for a review of the following topics:

- Random Variables: Chapters 2.4-2.6
- Normal Distribution: Chapters 3.1
- Linear Regression: Chapters 7.1-7.2

This refresher material is entirely optional, we hope you find it useful!

Diez et al. (2015, July): Some useful exercises

- Guided Practice 2.68–2.88
- Exercises 2.33—2.44 (odds have solutions)
- Guided Practice 3.1–3.23
- Exercises 3.1—3.16 (odds have solutions)
- Guided Practice 7.4–7.14
- Exercises 7.1–7.30 (odds have solutions)

Remarks:

- To test your ability, try solving a few problems from each section listed above
- Solutions to all guided practice and odd-numbered exercises are in the textbook

Lecture 01: Math Refresher (Optional Self-Study)

Revision Checklist

- ☐ Mathematical Prerequisites
- ☐ Pre-Calculus Refresher
- ☐ Calculus Refresher
- ☐ Statistics Refresher

Reading: Stitz and Zeager (2013, optional, Ch 1, 6, & 9), Hillier et al. (2016, optional, Ch 4 & 5), Hartman et al. (2018, optional, Ch 1 & 2), Diez et al. (2015, July, optional, Ch 2 & 7)

Intermediate Finance
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Lecture 02: Investment Under Certainty
Overview of Topics

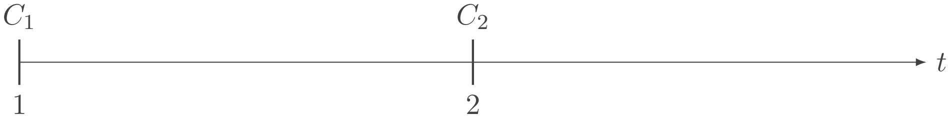
2.1. Intertemporal utility function

2.2. Intertemporal budget constraint

2.3. Capital investment and Fisher separation

Reading: Hillier et al. (2016, App 4A)

Investors plan sequences of consumption over time



Notation and remarks:

C_t Consumption in period t; we consider two periods: $t = 1$ and $t = 2$

- An investor chooses a sequence of consumption C_1 and C_2 in periods 1 and 2
- Assume that the investor faces no uncertainty with respect to the future
- How should the investor allocate consumption over time? What constraints apply?

Investors plan present and future consumption to maximize utility

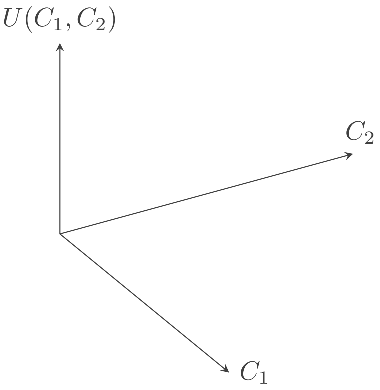
$$U(C_1, C_2)$$

Notation and remarks:

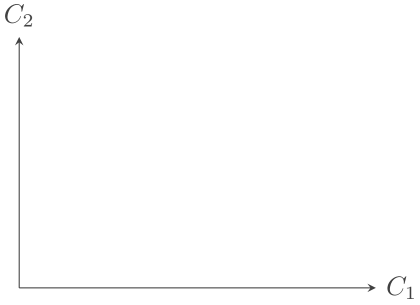
$U(\cdot)$ Intertemporal utility function, assigns utility value to consumption sequences

- Intertemporal utility describes investor preferences over consumption C_1 and C_2
- Investor objective: choose consumption C_1 and C_2 to maximize utility
- We assume utility strictly increases in each argument, holding the other fixed
- But we assume that utility increases in each argument at a diminishing rate
- Consumption choices are constrained by an intertemporal budget constraint

Drawing utility functions and indifference curves: a simple example



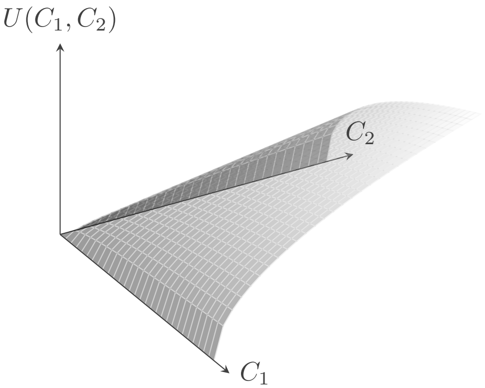
Indifference Curves

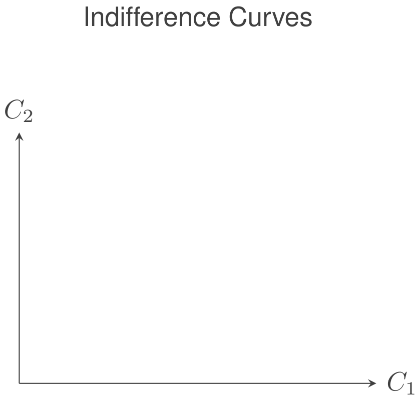


Remarks:

- The utility function at left shows utility values for consumption sequences (C_1, C_2)
- Indifference curves at right show sequences (C_1, C_2) that give the investor equal utility

Drawing utility functions and indifference curves: a simple example





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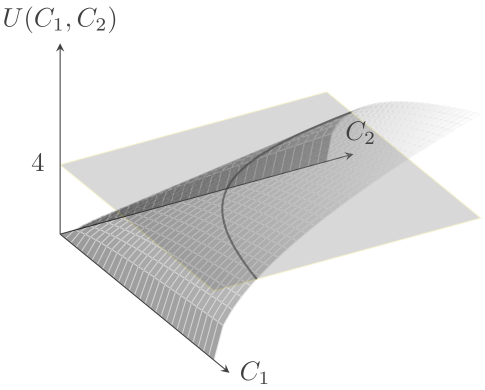

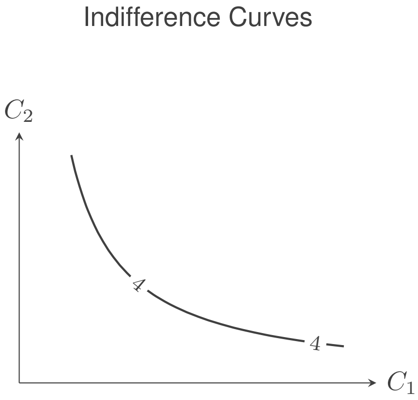

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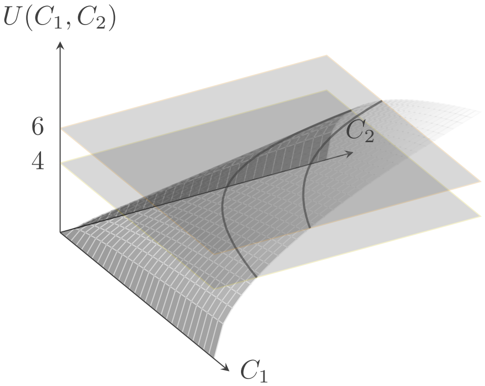

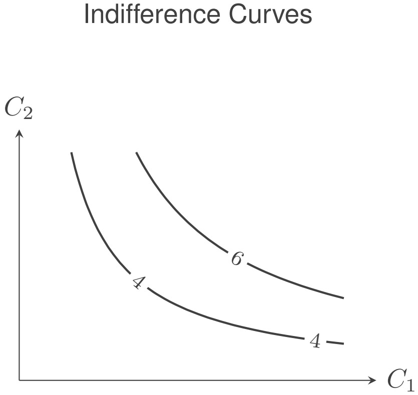

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Lecture 02: Investment Under Certainty
Overview of Topics

Reading: Hillier et al. (2016, App 4A)

An intertemporal budget constraint restricts consumption choices

$$C_1 + \frac{C_2}{1+R} \leq Y_1 + \frac{Y_2}{1+R}$$

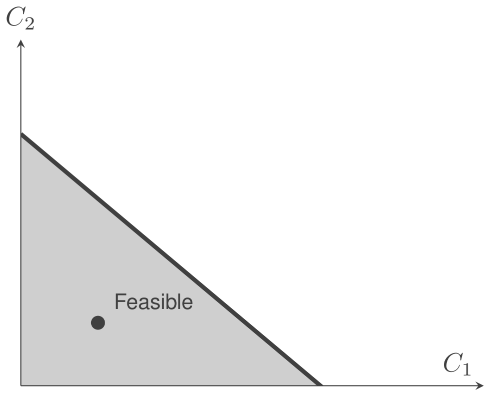
Notation and remarks:

Y_t Investor income in period t

R Discount rate, at which the investor can borrow or lend

- The IBC says PV(lifetime consumption) mustn't exceed PV(lifetime income)
- Must investors consume exactly their income each period (i.e. $C_1 = Y_1$, $C_2 = Y_2$)?
- No, investors can borrow or lend at rate R, to shift consumption across time

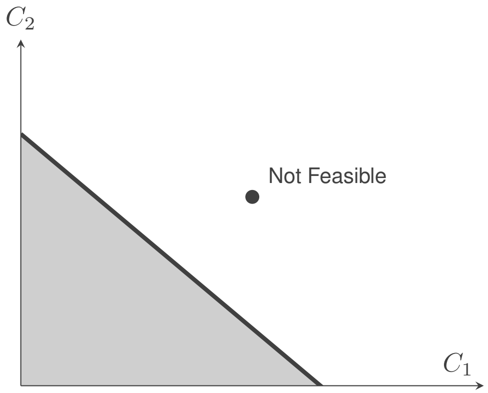
Any consumption sequence in the shaded region is feasible



Remarks:

- Line: IBC at equality
- Shade: feasible consumption

Any consumption sequence in the shaded region is feasible



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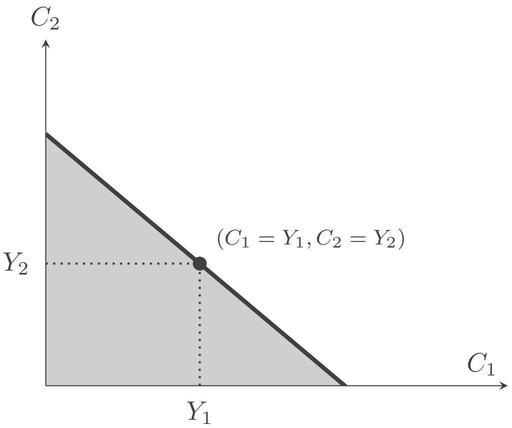

Remarks:

- Line: IBC at equality
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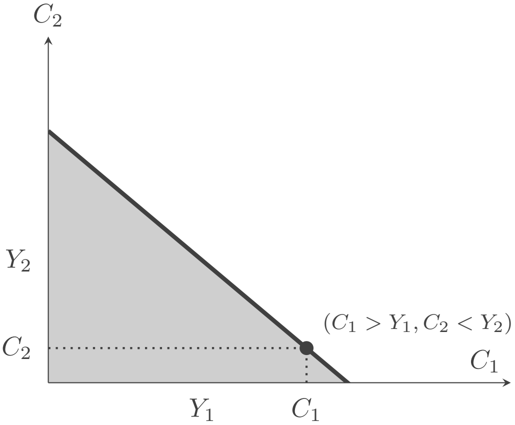
- Three feasible examples:
  1. Neither borrow nor lend: $C_1 = Y_1, C_2 = Y_2$

Any consumption sequence in the shaded region is feasible



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- Line: IBC at equality
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- Three feasible examples:
  1. Neither borrow nor lend: $C_1 = Y_1, C_2 = Y_2$
  2. Borrow funds today: $C_1 > Y_1, C_2 < Y_2$

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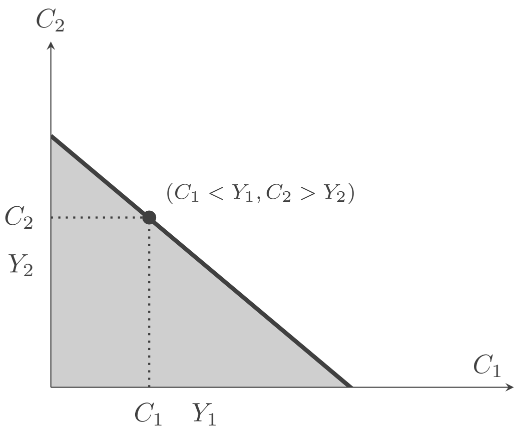

Remarks:

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- Three feasible examples:
  1. Neither borrow nor lend:
     $C_1 = Y_1, C_2 = Y_2$
  2. Borrow funds today:
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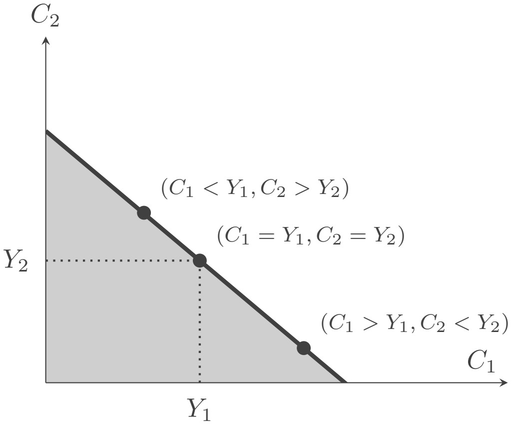
  3. Lend funds today:
     $C_1 < Y_1, C_2 > Y_2$

Any consumption sequence in the shaded region is feasible



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  2. Borrow funds today: $C_1 > Y_1, C_2 < Y_2$
  3. Lend funds today: $C_1 < Y_1, C_2 > Y_2$
- Which would you prefer?

Question 2

Suppose I offer you free coffee: the choice between 10 free cups this month, or 12 free cups next month. Which would you choose?

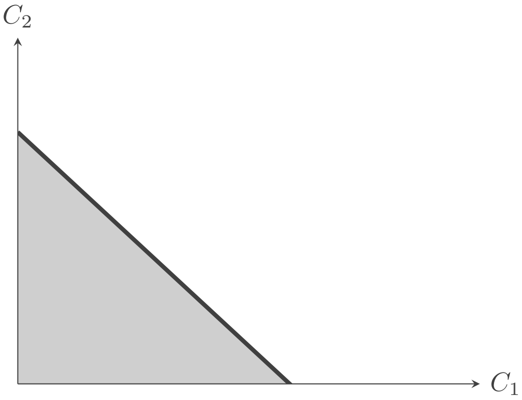
A. 10 free cups this month

B. 12 free cups next month

Remarks:

- Individuals often differ in their preferences over the timing of consumption
- In particular, different shareholders of a firm may have difference preferences
- Do differences in preferences matter for the firm's investment decisions?

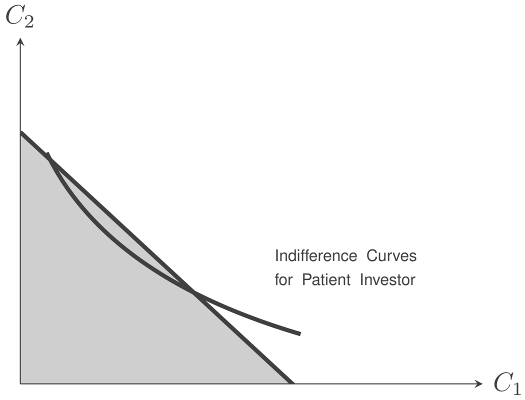
Different investors may prefer different consumption bundles



Remarks:

- IBC shows feasible consumption sequences
- Optimal consumption depends on IBC and utility

Different investors may prefer different consumption bundles



Remarks:

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- Patient investor ⇒ lower C_1 and higher C_2

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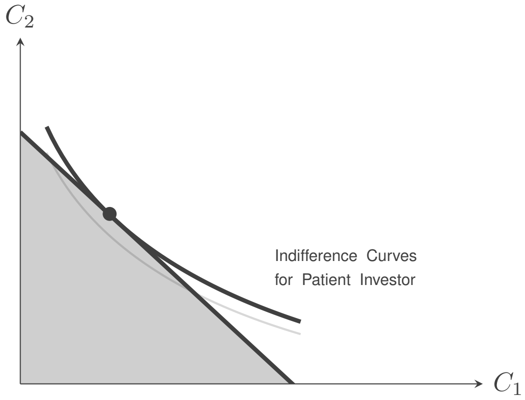

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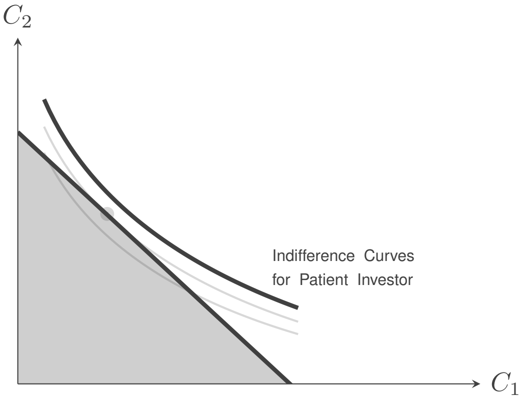

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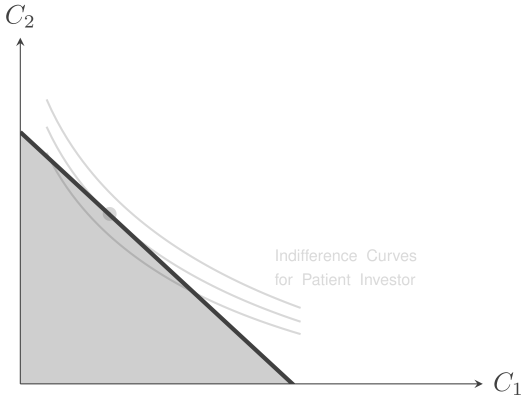

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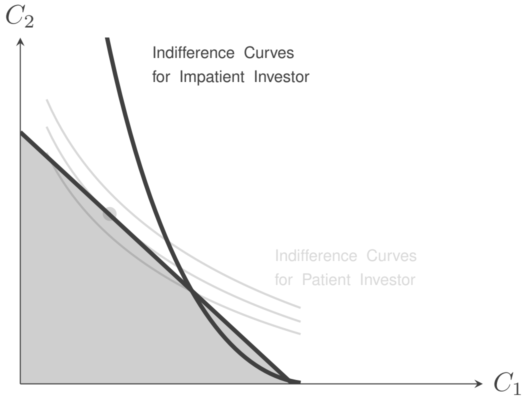

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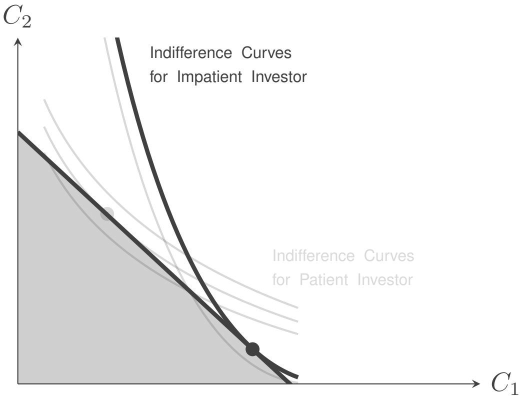
- Impatient investor $\Rightarrow$ higher C_1 and lower C_2

Different investors may prefer different consumption bundles



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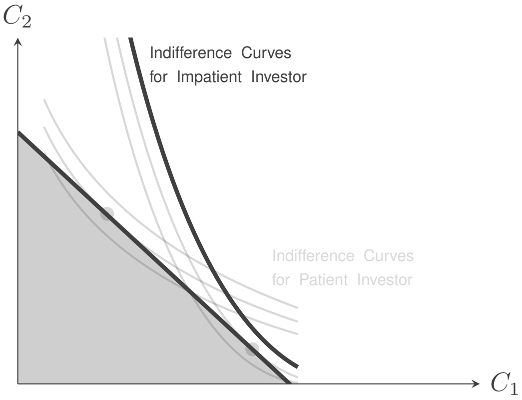

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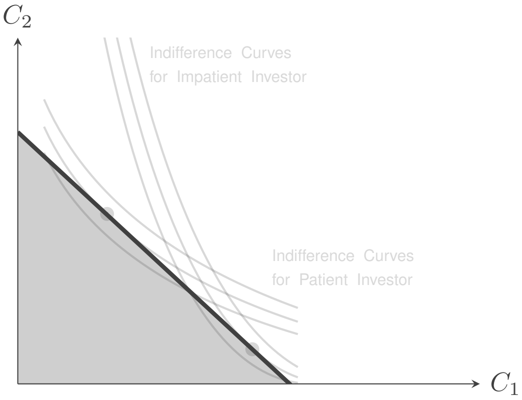
Different investors may prefer different consumption bundles



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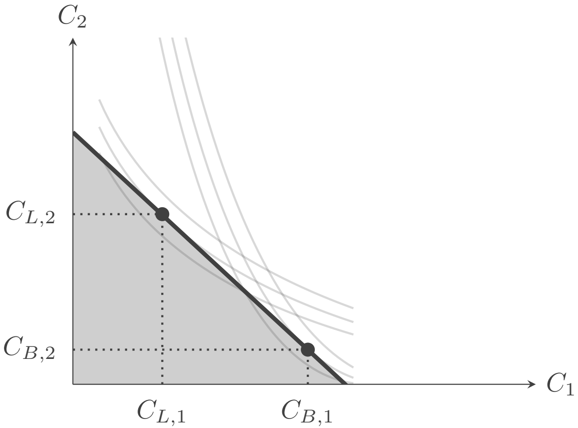

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- Patient investor $\Rightarrow$ lower C_1 and higher C_2
- Impatient investor $\Rightarrow$ higher C_1 and lower C_2
- Patient investor lends (L), impatient investor borrows (B)

Lecture 02: Investment Under Certainty
Overview of Topics

2.1. Intertemporal utility function

2.2. Intertemporal budget constraint

2.3. Capital investment and Fisher separation

Reading: Hillier et al. (2016, App 4A)

Intertemporal budget constraints can include investment projects

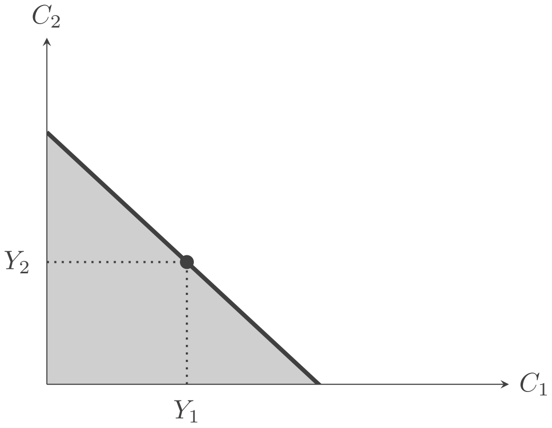
$$C_1 + \frac{C_2}{1+R} + I_1 \leq Y_1 + \frac{Y_2}{1+R} + \frac{CF_2}{1+R}$$

Notation and remarks:

- I_t Cost of investment project paid in period t
- CF_t Cash flow from investment project in period t
- R_p Project return, defined here as $R_p \coloneqq (CF_2 - I_1)/I_1$

- The investor can now invest I_1 in period 1 and receive CF_2 in period 2
- The firm should carry out the project if $NPV_1 = -I_1 + CF_2/(1+R) > 0$ holds
- Equivalently, the firm should carry out the project if $R_p > R$

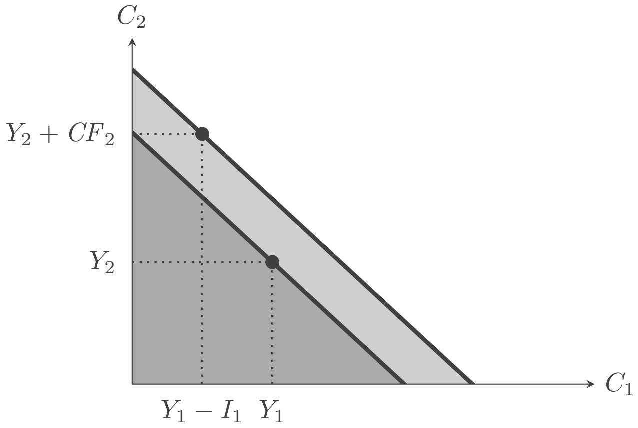
Intertemporal budget constraint with capital investment



Remarks:

- Suppose an investor consumed $C_1 = Y_1, C_2 = Y_2$ with no project

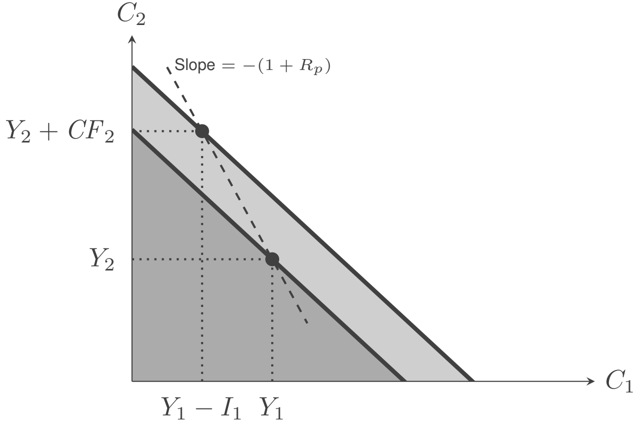
Intertemporal budget constraint with capital investment



Remarks:

- Suppose an investor consumed $C_1 = Y_1, C_2 = Y_2$ with no project
- Now with project, $Y_1 - I_1 < C_1$, $Y_2 + CF_2 > C_2$ is feasible

Intertemporal budget constraint with capital investment



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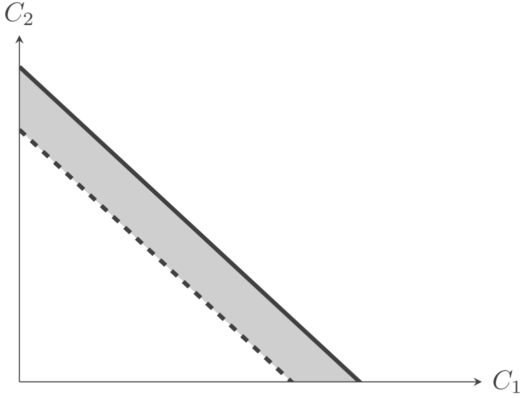
- Suppose an investor consumed $C_1 = Y_1, C_2 = Y_2$ with no project
- Now with project, $Y_1 - I_1 < C_1$, $Y_2 + CF_2 > C_2$ is feasible
- The project returns $R_p > R$, so the IBC shifts outward

Intertemporal budget constraint with capital investment



Remarks:

- Suppose an investor consumed $C_1 = Y_1, C_2 = Y_2$ with no project
- Now with project, $Y_1 - I_1 < C_1$, $Y_2 + CF_2 > C_2$ is feasible
- The project returns $R_p > R$, so the IBC shifts outward
- With project, better consumption sequences become feasible

The Fisher Separation Theorem

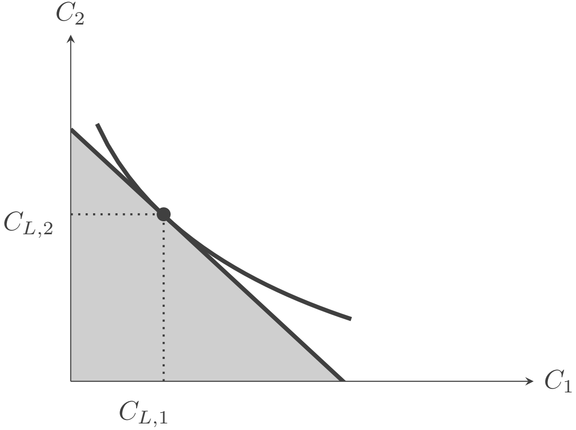
Definition 1

Fisher Separation Theorem	Absent asymmetries between borrowing and lending rates (and absent other imperfections), firms can make investment decisions independently of the intertemporal consumption preferences of investors.

Remarks:

- Consumption choices depend on preferences, which capture e.g. impatience
- Some investors may be very impatient—does this matter for capital budgeting?
- No, because borrowing and lending allow for any intertemporal consumption profile
- This result breaks however when borrowing and lending rates differ from each other
- Borrowing and lending rates commonly differ: e.g. savings accounts vs. credit cards

The project makes patient and impatient investors better off



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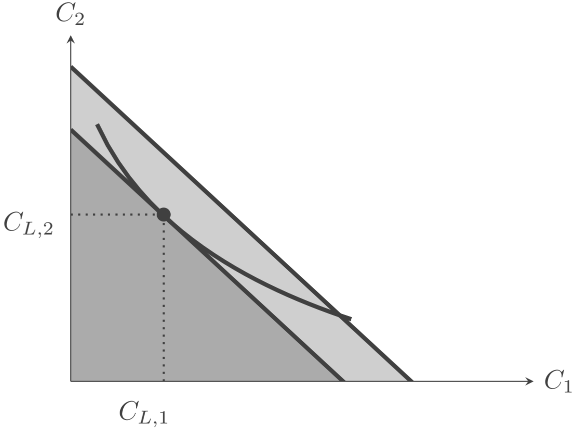

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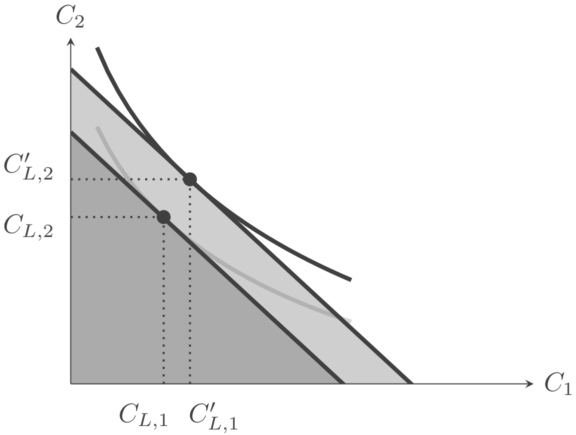
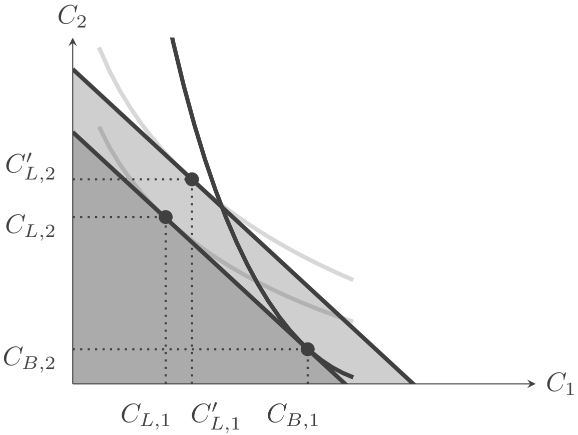


Remarks:

- Project allows patient investor to move to higher indifference curve

The project makes patient and impatient investors better off



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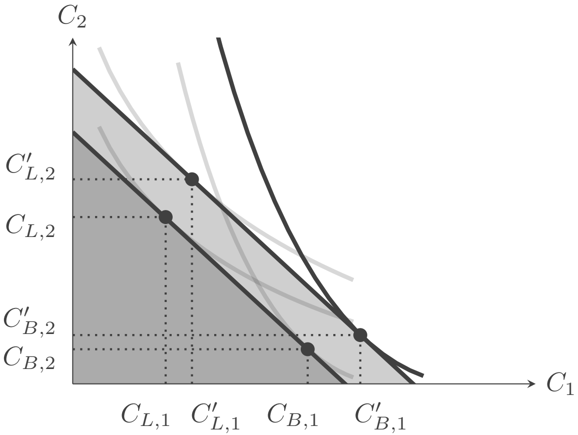

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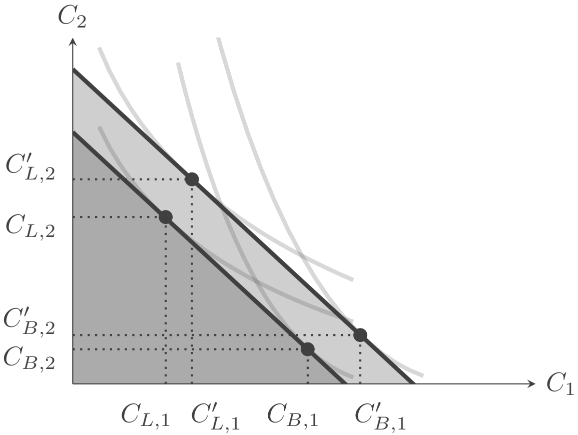
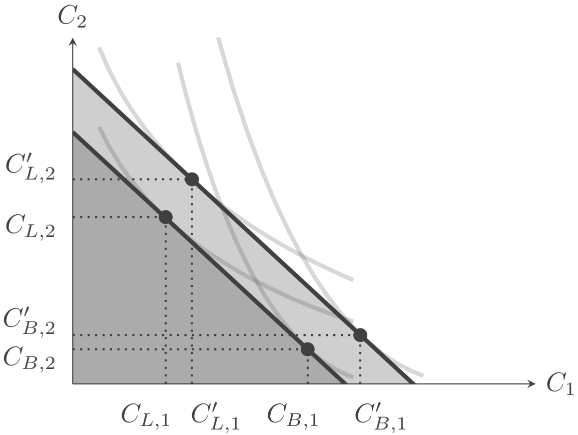
The project makes patient and impatient investors better off



Remarks:

- Project allows patient investor to move to higher indifference curve
- Project allows impatient investor to move to higher indifference curve
- Because all investors better off, firm should carry out investment project

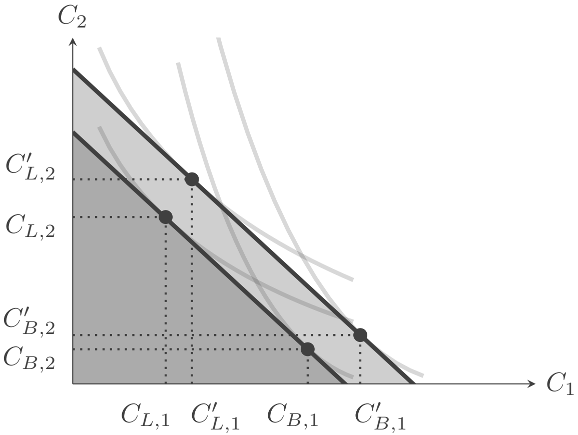
The project makes patient and impatient investors better off



Remarks:

- Project allows patient investor to move to higher indifference curve
- Project allows impatient investor to move to higher indifference curve
- Because all investors better off, firm should carry out investment project
- Result depends crucially on project return R_p exceeding rate R

The project makes patient and impatient investors better off



Remarks:

- Project allows patient investor to move to higher indifference curve
- Project allows impatient investor to move to higher indifference curve
- Because all investors better off, firm should carry out investment project
- Result depends crucially on project return R_p exceeding rate R
- Result also depends crucially on ability to borrow *and* lend at rate R

Lecture 02: Investment Under Certainty

Revision Checklist

- ☐ Intertemporal utility function
- ☐ Intertemporal budget constraint
- ☐ Capital investment and Fisher separation

Reading: Hillier et al. (2016, App 4A)

Intermediate Finance
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Lecture 03: Risk and Expected Return
Overview of Topics

Reading: Hillier et al. (2016, Ch 9), Bodie et al. (2014, Ch 5 & 18)

Some sharper definitions will help us think about randomness

Definition 2

Random Process:	A situation in which we know what outcomes could happen, but we don't know which outcome will happen.
Random Variable:	A numeric value that depends on the realized outcome of a random process.

Remarks:

- Returns are random because we can't predict with certainty their future values
- To deal with this uncertainty, we next introduce to the concept of probability

Two rules for working with probabilities, and three interpretations

Definition 3

Rule 1	The probability of any outcome must lie between 0 and 1.
Rule 2	The probabilities of all possible outcomes must sum to 1.

Remarks:

- Classical view: outcomes are uncertain, but probabilities of outcomes are known with certainty; we mostly take the classical view in these lectures
- Frequentist view: Probability may be unknown, but is estimated as the proportion of times an outcome would occur if a random process were repeated infinitely
- Bayesian view: Probability is a subjective degree of belief, and must be updated rationally whenever new information arrives using Bayes' theorem

Lecture 03: Risk and Expected Return
Overview of Topics

Reading: Hillier et al. (2016, Ch 9), Bodie et al. (2014, Ch 5 & 18)

Discrete random variables take countably many possible values

Definition 4

Discrete Random Variable	A numeric quantity, dependent on the realization of the outcome of a random process, that takes countably many possible values.

Remarks:

- Possible values of discrete random variables can be counted 1-to-1 with the integers
- Example: a coin toss with X equals 1 if heads H and X equals 0 if tails T
- The coin toss is a random process, and $\{H, T\}$ are the possible outcomes
- Random variable X takes values $\{0, 1\}$ depending on realized outcome

Expected value and variance help us characterize of uncertainty

Expected Value: $$\mu_X = \mathrm{E}[X] = \sum_{n=1}^{N} p_n x_n$$

Variance: $$\sigma_X^2 = \mathrm{Var}(X) = \mathrm{E}\Big[(X - \mu_X)^2\Big] = \sum_{n=1}^{N} p_n (x_n - \mu_X)^2$$

Notation:

X	Discrete random variable, with N possible realizations x_n, where $n \in \{1, 2, \ldots, N\}$
p_n	Probability that X takes realization x_n, also denoted $\mathrm{Prob}(X = x_n)$ or $p(x_n)$
μ_X	Expected value of random variable X, a measure of central tendency
σ_X^2	Variance of random variable X, a measure of dispersion; $\sigma_X = \sqrt{\sigma_X^2}$ is std deviation
$\mathrm{E}[\cdot]$	Expectation operator
$\mathrm{Var}(\cdot)$	Variance operator

Learn these useful rules for working with variance

Difference Rule: $\text{Var}(X) = \text{E}\left[X^2\right] - \text{E}[X]^2$

Scalar Rule: $\text{Var}(aX) = a^2\text{Var}(X)$

Sum Rule: $\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X, Y)$

Notation and remarks:

aX, bY Random variables X, Y multiplied by non-random scalar coefficients a, b

- These rules can save you time and simplify calculations if you memorize them
- Covariance $\text{Cov}(\cdot)$ measures comovement between two random variables
- Measuring comovement is crucial in finance, so we discuss if further below

Lecture 03: Risk and Expected Return
Overview of Topics

Reading: Hillier et al. (2016, Ch 9), Bodie et al. (2014, Ch 5 & 18)

Covariance and correlation between two random variables

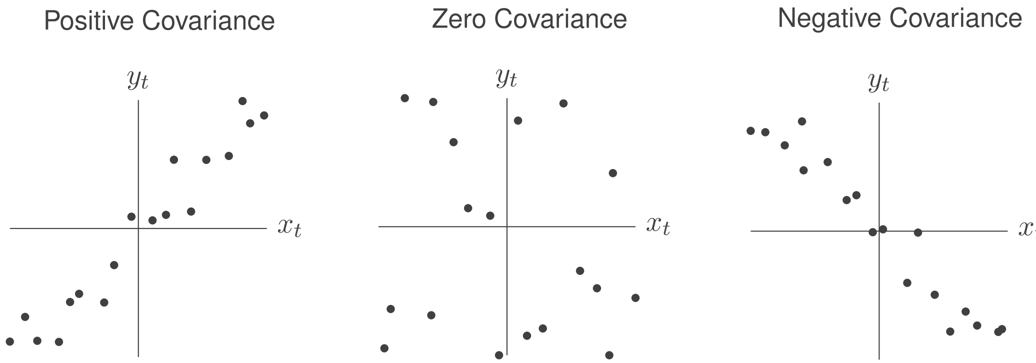
Covariance: $$\sigma_{XY} = \mathrm{Cov}(X,Y) = \mathrm{E}\big[(X-\mu_X)(Y-\mu_Y)\big] = \sum_{n=1}^{N} p_n(x_n-\mu_X)(y_n-\mu_Y)$$

Correlation: $$\rho_{XY} = \mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Sd}(X)\mathrm{Sd}(Y)}, \quad \rho_{XY} \in [-1,1]$$

Notation and remarks:

$\mathrm{Cov}(\cdot,\cdot)$	Covariance operator, measures comovement between random variables
$\mathrm{Corr}(\cdot,\cdot)$	Correlation operator, normalizes covariance, takes values in $[-1,1]$
$\mathrm{Sd}(\cdot)$	Standard deviation operator, $\mathrm{Sd}(X) := \sqrt{\mathrm{Var}(X)}$

Covariance has a simple and intuitive geometric interpretation



Remarks:

- Let (x_t, y_t) be monthly realizations of your wage income X and investment income Y
- How would you prefer your wage income to covary with your investment income?

Learn these useful rules for working with covariance

Difference Rule: $$\mathrm{Cov}(X, Y) = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]$$

Scalar Rule: $$\mathrm{Cov}(aX, bY) = ab\mathrm{Cov}(X, Y)$$

Sum Rule: $$\mathrm{Cov}(aX_1 + bX_2, Y) = a\mathrm{Cov}(X_1, Y) + b\mathrm{Cov}(X_2, Y)$$

Notation and remarks:

aX, bY Random variables X, Y multiplied by non-random scalar coefficients a, b

- These rules can save you time and simplify calculations if you memorize them

Lecture 03: Risk and Expected Return
Overview of Topics

Reading: Hillier et al. (2016, Ch 9), Bodie et al. (2014, Ch 5 & 18)

Numerical Example: Computing Expected Returns

Question 3

Returns on car shares and gold are given in the table below for three equally-probable states of the economy:

State	Returns on:	Cars	Gold
Recession		-8%	20%
Normal		5%	3%
Growth		18%	-20%

What are the expected returns on car shares and gold?

A. $\mathrm{E}[R_c] = 1\%$, $\mathrm{E}\left[R_g\right] = 5\%$

B. $\mathrm{E}[R_c] = 5\%$, $\mathrm{E}\left[R_g\right] = 1\%$

Numerical Example: Computing Expected Returns

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What are the expected returns on car shares and gold?

A. $\mathrm{E}[R_c] = 1\%, \mathrm{E}\left[R_g\right] = 5\%$

B. $\mathbf{E}[\boldsymbol{R_c}] = \mathbf{5\%}, \mathbf{E}\left[\boldsymbol{R_g}\right] = \mathbf{1\%}$

Numerical Example: Computing Expected Returns

Solution 3

The states are equally probable, so $p_n = 1/3$ for all n.

Expected returns on car shares are therefore

$$\mu_c = \mathrm{E}[R_c] = \frac{1}{3}(-8\%) + \frac{1}{3}(5\%) + \frac{1}{3}(18\%) = 5\%\,.$$

Expected returns on gold are therefore

$$\mu_g = \mathrm{E}[R_g] = \frac{1}{3}(20\%) + \frac{1}{3}(3\%) + \frac{1}{3}(-20\%) = 1\%\,.$$

Numerical Example: Computing Variance of Returns

Question 4

Returns on car shares and gold are given in the table below for three equally-probable states of the economy:

State	Returns on:	Cars	Gold
Recession		-8%	20%
Normal		5%	3%
Growth		18%	-20%

What are the variances of returns on car shares and gold?

A. $\mathrm{Var}(R_c) = 112.7\%\%$, $\mathrm{Var}\left(R_g\right) = 268.7\%\%$

B. $\mathrm{Var}(R_c) = 121.7\%\%$, $\mathrm{Var}\left(R_g\right) = 286.7\%\%$

Numerical Example: Computing Variance of Returns

Question 4

Returns on car shares and gold are given in the table below for three equally-probable states of the economy:

State	Returns on:	Cars	Gold
Recession		-8%	20%
Normal		5%	3%
Growth		18%	-20%

What are the variances of returns on car shares and gold?

A. $\mathbf{Var(R_c) = 112.7\%\%,\ Var\big(R_g\big) = 268.7\%\%}$

B. $\mathrm{Var}(R_c) = 121.7\%\%,\ \mathrm{Var}\big(R_g\big) = 286.7\%\%$

Numerical Example: Computing Variance of Returns

Solution 4

The states are equally probable, so $p_n = 1/3$ for all n.

Variance of returns on car shares are therefore

$$\sigma_c^2 = \mathrm{Var}(R_c) = \frac{1}{3}(-8\% - 5\%)^2 + \frac{1}{3}(5\% - 5\%)^2 + \frac{1}{3}(18\% - 5\%)^2 = 112.7\%\%\,.$$

Variance of returns on gold are therefore

$$\sigma_g^2 = \mathrm{Var}(R_g) = \frac{1}{3}(20\% - 1\%)^2 + \frac{1}{3}(3\% - 1\%)^2 + \frac{1}{3}(-20\% - 1\%)^2 = 268.7\%\%\,.$$

Numerical Example: Computing Covariance of Returns

Question 5

Returns on car shares and gold are given in the table below for three equally-probable states of the economy:

State	Returns on:	Cars	Gold
Recession		-8%	20%
Normal		5%	3%
Growth		18%	-20%

What is the covariance between returns on car shares and gold?

A. $\mathrm{Cov}(R_c, R_g) = \quad 137.33\%\%$

B. $\mathrm{Cov}(R_c, R_g) = -173.33\%\%$

Numerical Example: Computing Covariance of Returns

Question 5

Returns on car shares and gold are given in the table below for three equally-probable states of the economy:

State	Returns on:	Cars	Gold
Recession		-8%	20%
Normal		5%	3%
Growth		18%	-20%

What is the covariance between returns on car shares and gold?

A. $\mathrm{Cov}(R_c, R_g) = \quad 137.33\%\%$

B. $\mathbf{Cov}(\boldsymbol{R_c}, \boldsymbol{R_g}) = \mathbf{-173.33\%\%}$

Numerical Example: Computing Covariance of Returns

Solution 5

The states are equally probable, so $p_n = 1/3$ for all n.

The covariance of returns on car shares are therefore

$$\begin{aligned}\mathrm{Cov}(R_c, R_g) &= \frac{1}{3}(-8\% - 5\%)(20\% - 1\%)\\ &+ \frac{1}{3}(5\% - 5\%)(3\% - 1\%)\\ &+ \frac{1}{3}(18\% - 5\%)(-20\% - 1\%) = -173.33\%\% \,.\end{aligned}$$

Numerical Example: Computing Normalized Statistics

Question 6

Returns on car shares and gold are given in the table below for three equally-probable states of the economy:

State	Returns on:	Cars	Gold
Recession		-8%	20%
Normal		5%	3%
Growth		18%	-20%

What are the standard deviations of returns and correlation between returns on car shares and gold?

A. $\sigma_c = 10.61\%$, $\sigma_g = 16.39\%$, $\rho_{cg} = -100\%$

B. $\sigma_c = 16.39\%$, $\sigma_g = 10.61\%$, $\rho_{cg} = 100\%$

Numerical Example: Computing Normalized Statistics

Question 6

Returns on car shares and gold are given in the table below for three equally-probable states of the economy:

State	Returns on:	Cars	Gold
Recession		-8%	20%
Normal		5%	3%
Growth		18%	-20%

What are the standard deviations of returns and correlation between returns on car shares and gold?

A. $\boldsymbol{\sigma_c = 10.61\%, \sigma_g = 16.39\%, \rho_{cg} = -100\%}$

B. $\sigma_c = 16.39\%, \sigma_g = 10.61\%, \rho_{cg} = 100\%$

Numerical Example: Computing Normalized Statistics

Solution 6

The standard deviations are the square roots of the variances computed above, so

$$\sigma_c = \mathrm{Sd}(R_c) = \sqrt{112.7\%\%} = 10.61\%$$
$$\sigma_g = \mathrm{Sd}(R_g) = \sqrt{268.7\%\%} = 16.39\%\,.$$

The correlation of returns equals the covariance of returns divided by the product of standard deviations:

$$\rho_{cg} = \mathrm{Corr}(R_c, R_g) = -173.33\%\%/(10.61\% \cdot 16.39\%) = -100\%\,.$$

Returns are perfectly negatively correlated.

Lecture 03: Risk and Expected Return
Overview of Topics

Reading: Hillier et al. (2016, Ch 9), Bodie et al. (2014, Ch 5 & 18)

Some random variables take *un*-countably many possible values

Definition 5

Continuous Random Variable	A numeric quantity, dependent on the realization of the outcome of a random process, that takes uncountably many possible values.

Remarks:

- Possible values of continuous random variables cannot be counted by the integers
- Example: your exact height X next year, a positive real number, i.e. $X \in \mathbb{R}^+$

Expected value and variance of continuous random variables

Expected Value: $$\mu_X = \mathrm{E}[X] = \int_{-\infty}^{+\infty} p(x) x \,\mathrm{d}x$$

Variance: $$\sigma_X^2 = \mathrm{Var}(X) = \mathrm{E}\Big[(X - \mu_X)^2\Big] = \int_{-\infty}^{+\infty} p(x)(x - \mu_X)^2 \,\mathrm{d}x$$

Notation and remarks:

- X Continuous random variable, with realizations $x \in \mathbb{R}$
- $p(x)$ Probability that X takes a value in small interval dx around x
- Examples of continuous distributions: Normal distribution, T-distribution
- Because uncountably many possible values, probability of a point value is zero
- The probability that the realization lies in an interval dx is more meaningful here

Lecture 03: Risk and Expected Return

Revision Checklist

- ☐ Random variables
- ☐ Discrete Random Variables: Mean and Variance
- ☐ Discrete Random Variables: Comovement
- ☐ Discrete Random Variables: Numerical Examples
- ☐ Continuous Random Variables

Reading: Hillier et al. (2016, Ch 9), Bodie et al. (2014, Ch 5 & 18)

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Lecture 04: Risk Aversion and Expected Utility I
Overview of Topics

4.1. Risk and uncertainty

4.2. Utility and Risk Aversion

4.3. Expected wealth and utility

Reading: Bodie et al. (2014, Ch 6)

Risk: a loose definition

Definition 6

Risk	A quantifiable measure of the degree of uncertainty surrounding a random variable before its value is realized.

Remarks:

- We quantify risk with statistical measures of dispersion like variance and std deviation
- Attitudes towards risk are captured by the shape of an investor's utility function
- The attitude of risk aversion is particularly important in finance and economics
- Risk averse investors demand compensation for accepting risk—a risk premium

Note: Here, outcomes are uncertain, but probabilities are not. Cf. the distinction between risk and uncertainty that Knight (1921) makes.

Actuarially fair gambles are useful for thinking about risk-taking

Definition 7

Actuarially Fair Gamble	A risky gamble with probabilities and payoffs chosen such that the expected value of the gamble equals zero.

Remarks:

- Suppose a gamble X has payoffs x_1, x_2 with probabilities π, $1-\pi$, respectively
- Then the gamble X is actuarially fair if for profit X, $\mathrm{E}[X] = \pi x_1 + (1-\pi)x_2 = 0$
- Term “actuarially fair” comes from insurance: fair if premium equals expected claim
- Are real insurance contracts actuarially fair? (Think market power, scale economies)

A simple example: a free and fair coin toss

Heads: Investor gains £50

Tails: Investor loses £50

Remarks:

- Expected future payoff: $\mathrm{E}[X] = 0.5 \cdot (+50) + 0.5 \cdot (-50) = \pounds 0$
- The gamble is clearly actuarially fair—but should an investor take the gamble?
- Answer depends on investor's attitude towards risk (the shape of their utility function)
- We'll later see: risk-neutral $\Rightarrow$ indifferent to gamble; risk-averse $\Rightarrow$ reject the gamble

Lecture 04: Risk Aversion and Expected Utility I
Overview of Topics

Reading: Bodie et al. (2014, Ch 6)

Question 7

I’m offering those with new iPhones the following risky gamble: We flip a coin. If we get heads I keep your iPhone and mine. If we get tails you keep your iPhone and mine. This is a serious offer. Any takers?

A. Yes, I’m in.

B. No, I’m out.

Utility over wealth for risk-averse investors

$$U(W), \qquad \text{with} \qquad U'(W) > 0, \qquad U''(W) < 0$$

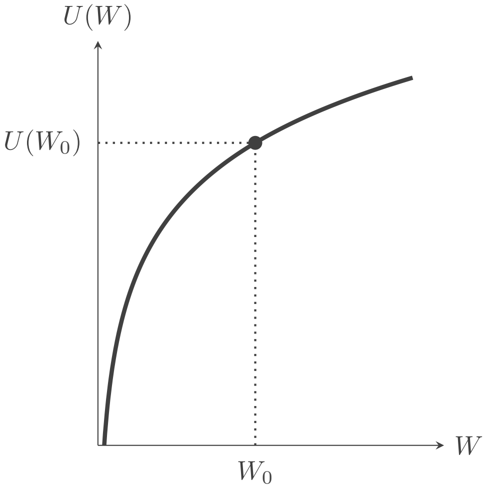
Notation and remarks:

W Wealth

$U(\cdot)$ Intratemporal utility over wealth, with derivatives $U'(\cdot)$ and $U''(\cdot)$

- Note: here utility is *intra*-temporal, unlike utility-over-consumption from last lecture
- Non-satiation: investors prefer more wealth to less wealth ($U'(W) > 0$)
- Diminishing marginal utility: poor gain more than rich as wealth rises ($U''(W) < 0$)
- Risk aversion: losses hurt more than gains help ($U''(W) < 0$)

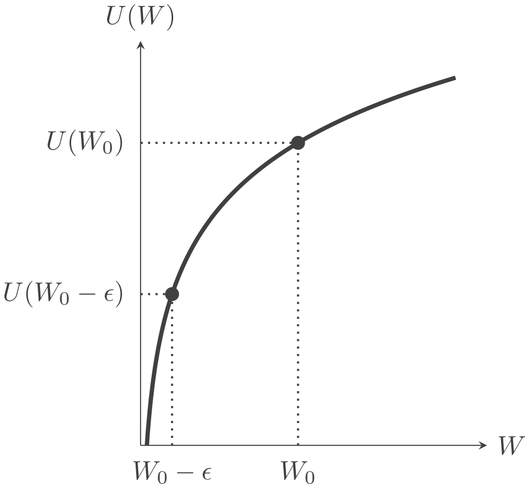
Risk aversion: losses hurt more than gains help



Remarks:

- Consider gains and losses $\pm\epsilon$ in initial wealth W_0

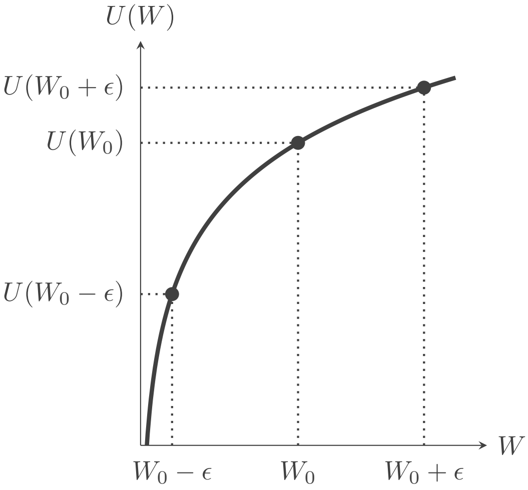
Risk aversion: losses hurt more than gains help



Remarks:

- Consider gains and losses $\pm\epsilon$ in initial wealth W_0
- A wealth loss $-\epsilon$ causes a relatively large drop in utility

Risk aversion: losses hurt more than gains help



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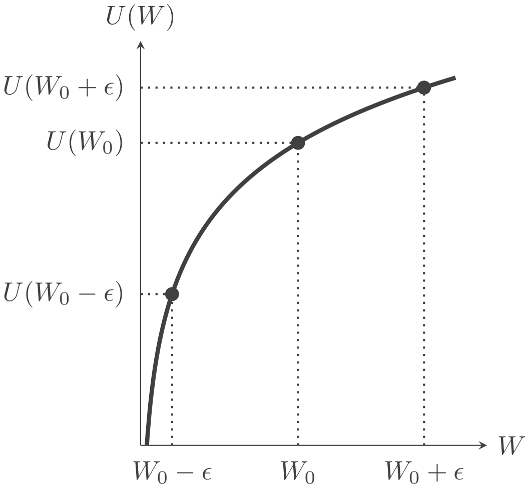


Remarks:

- Consider gains and losses $\pm\epsilon$ in initial wealth W_0
- A wealth loss $-\epsilon$ causes a relatively large drop in utility
- A wealth gain $+\epsilon$ causes a relatively small rise in utility
- The utility function is concave, so the investor is risk averse
- For risk averse investors, loss hurts more than gain helps

Measuring absolute and relative risk aversion

$$ARA = -\frac{U''(W)}{U'(W)}, \qquad RRA = -W\frac{U''(W)}{U'(W)}$$

Notation and remarks:

ARA Absolute risk aversion, an absolute measure of investor risk attitudes

RRA Relative risk aversion, a relative measure of investor risk attitudes

- Positive values $\Rightarrow$ risk averse, negative values $\Rightarrow$ risk-seeking
- If ARA falls in wealth, then wealth $\uparrow \Rightarrow$ dollar amount in risky assets $\uparrow$
- If RRA falls in wealth, then wealth $\uparrow \Rightarrow$ fraction of wealth in risky assets $\uparrow$
- Verify that $U(W) = -e^{-aW}$ satisfies constant absolute risk aversion
- Verify that $U(W) = (W^{1-a})/(1-a)$ satisfies constant relative risk aversion

Lecture 04: Risk Aversion and Expected Utility I
Overview of Topics

Reading: Bodie et al. (2014, Ch 6)

Expected wealth: a simple example

An investor with initial wealth W_0 faces an actuarially fair gamble with possible outcomes ϵ_1 and ϵ_2 and probabilities π and $1 - \pi$, respectively:

$$W = \begin{cases} W_0 + \epsilon_1, & \text{with probability } \pi, \\ W_0 + \epsilon_2, & \text{with probability } 1 - \pi. \end{cases}$$

Thus, expected wealth is unchanged by the gamble:

$$\begin{aligned} \mathrm{E}[W] &= \pi(W_0 + \epsilon_1) + (1 - \pi)(W_0 + \epsilon_2) \\ &= W_0 + \pi\epsilon_1 + (1 - \pi)\epsilon_2 \\ &= W_0. \end{aligned}$$

Expected utility of wealth: a simple example

The investor's utility of wealth after the gamble is uncertain:

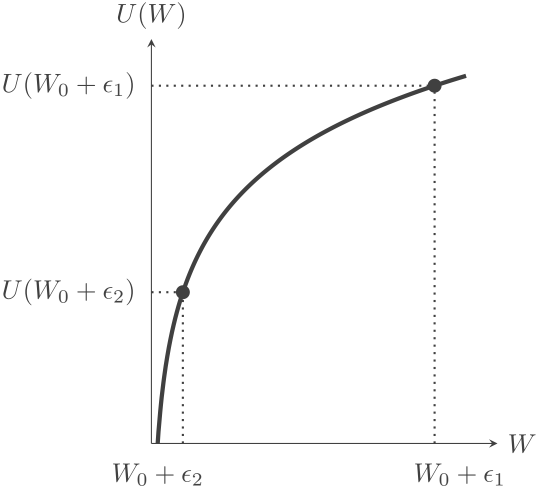
$$U(W) = \begin{cases} U(W_0 + \epsilon_1), & \text{with probability } \pi. \\ U(W_0 + \epsilon_2), & \text{with probability } 1 - \pi. \end{cases}$$

Thus, we can compute an expected utility of wealth:

$$\mathrm{E}\big[U(W)\big] = \pi U(W_0 + \epsilon_1) + (1 - \pi) U(W_0 + \epsilon_2)$$

Note well: For risk averse investors, $\mathrm{E}\big[U(W)\big] < U\big(\mathrm{E}[W]\big)$.

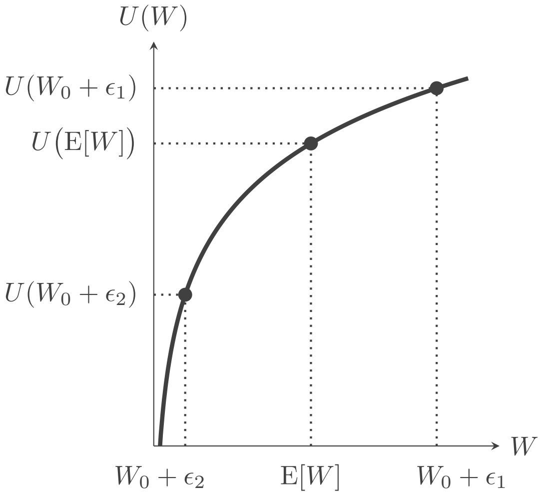
Utility of expected wealth versus expected utility of wealth



Remarks:

- Risk aversion implies concavity of utility over wealth
- Expected utility of wealth lies on the expected utility line
- Where exactly on the expected utility line depends on π
- For any probability $\pi \in (0, 1)$, $\mathrm{E}\big[U(W)\big] < U\big(\mathrm{E}[W]\big)$

Utility of expected wealth versus expected utility of wealth



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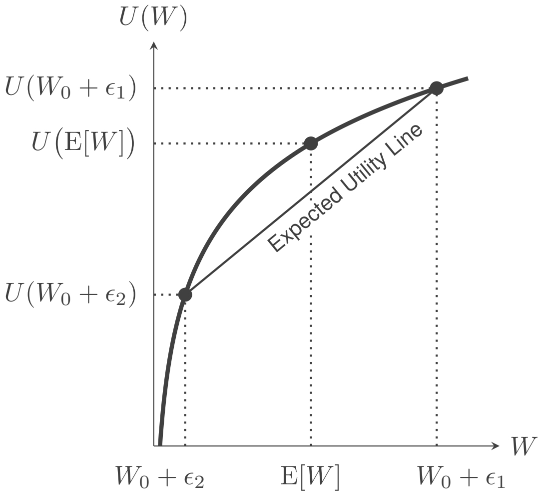

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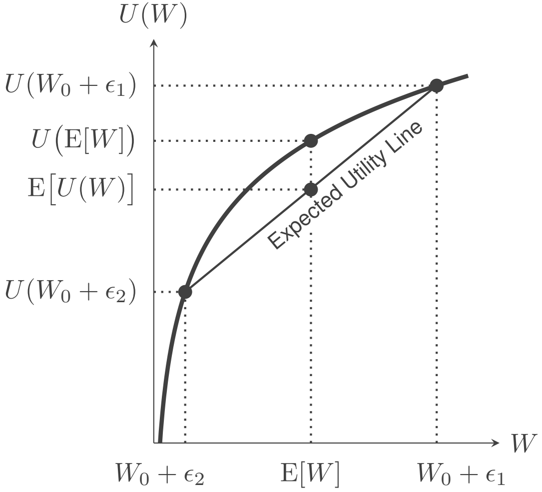
Utility of expected wealth versus expected utility of wealth



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Lecture 04: Risk Aversion and Expected Utility I

Revision Checklist

- ☐ Risk and uncertainty
- ☐ Utility and Risk Aversion
- ☐ Expected wealth and utility

Reading: Bodie et al. (2014, Ch 6)

Intermediate Finance
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Lecture 05: Risk Aversion and Expected Utility II
Overview of Topics

Reading: Bodie et al. (2014, Ch 6)

The certainty-equivalent level of wealth CE

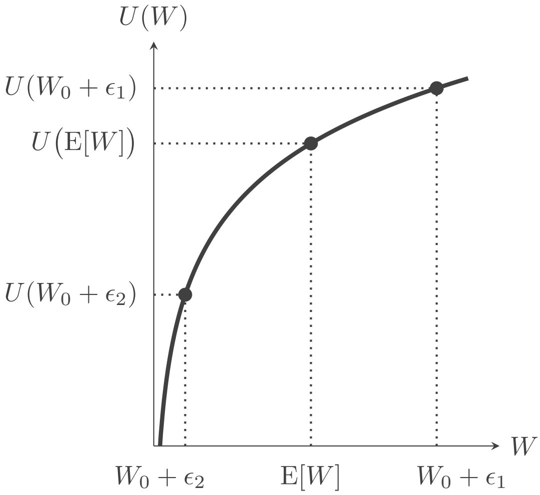
$$U(CE) = \mathrm{E}[U(W)]$$

Notation and remarks:

CE A level of *wealth* that yields with certainty the expected utility of a risky lottery

- A gambler is indifferent between the CE and the risky gamble on a better outcome
- Risk-aversion $\Rightarrow \mathrm{E}\big[U(W)\big] < U\big(\mathrm{E}[W]\big) \Rightarrow U(CE) < U\big(\mathrm{E}[W]\big) \Rightarrow CE < \mathrm{E}[W]$
- In other words, expected wealth exceeds the CE for risk-averse investors

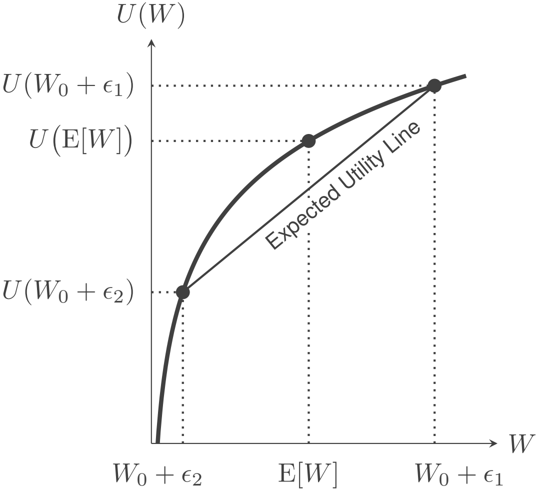
Finding the certainty-equivalent level of wealth geometrically



Remarks:

- Consider the concave utility of a risk-averse investor
- Initial wealth W_0 changes by amount $\epsilon_1 > 0$ or $\epsilon_2 < 0$

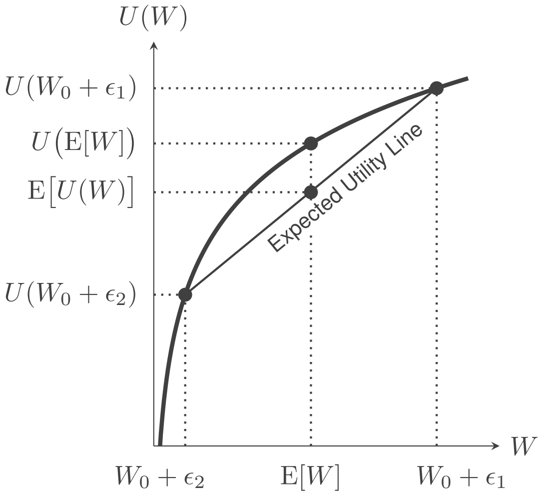
Finding the certainty-equivalent level of wealth geometrically



Remarks:

- Consider the concave utility of a risk-averse investor
- Initial wealth W_0 changes by amount $\epsilon_1 > 0$ or $\epsilon_2 < 0$
- Expected utility lies somewhere on the line shown at left

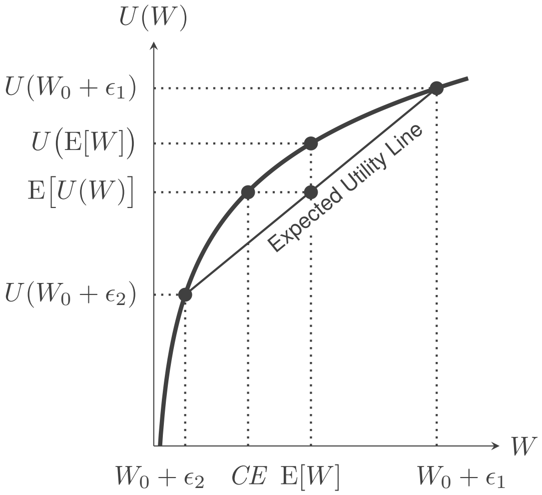
Finding the certainty-equivalent level of wealth geometrically



Remarks:

- Consider the concave utility of a risk-averse investor
- Initial wealth W_0 changes by amount $\epsilon_1 > 0$ or $\epsilon_2 < 0$
- Expected utility lies somewhere on the line shown at left
- Point $\Big(\mathrm{E}[W], \mathrm{E}\big[U(W)\big]\Big)$ thus sits on the line shown at left

Finding the certainty-equivalent level of wealth geometrically



Remarks:

- Consider the concave utility of a risk-averse investor
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- Point $\Big(\mathrm{E}[W], \mathrm{E}\big[U(W)\big]\Big)$ thus sits on the line shown at left
- CE is the level of wealth such that $U(CE) = \mathrm{E}\big[U(W)\big]$

Lecture 05: Risk Aversion and Expected Utility II
Overview of Topics

Reading: Bodie et al. (2014, Ch 6)

The Markowitz risk premium π_M

$$\pi_M = \mathrm{E}[W] - CE$$

Notation and remarks:

π_M Markowitz risk premium

- π_M is positive for risk-averse investors, who will pay a premium to avoid risk
- notice that π_M also satisfies $\mathrm{E}\big[U(W)\big] = U(W_0 - \pi_M)$ for actuarially fair gambles
- thus π_M is an upper bound on the price an investor would pay to avoid the gamble

Markowitz risk premium: a simple example

Question 8

You and your friend have phones worth \$500 each. You have a \$50 phone case, your friend has \$50 earbuds. You have no other wealth. Your friend wants to toss a coin: heads they win your phone case, tails you win their earbuds. If your utility is logarithmic, what would you pay to avoid this coin toss?

A. about \$2.50

B. about \$5.00

C. about \$10.00

Markowitz risk premium: a simple example

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Markowitz risk premium: a simple example

Solution 8

You have $W_0 = 550$, and the coin toss changes your wealth by $\epsilon_1 = 50$ or $\epsilon_2 = -50$, each with probability $\pi = 0.5$. Therefore,

$$W = \begin{cases} 600 & \text{with probability } 0.5 \\ 500 & \text{with probability } 0.5\,. \end{cases}$$

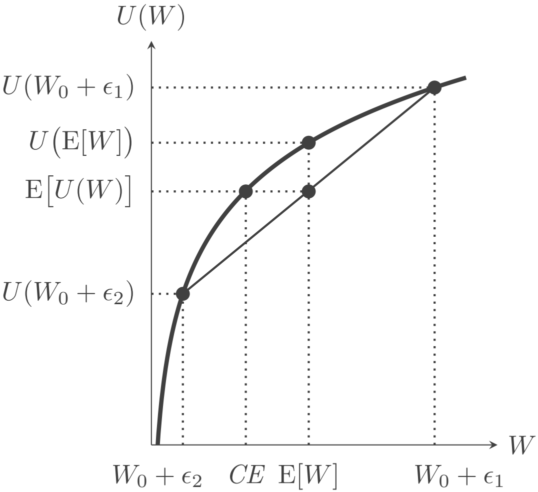
Your utility is $U(W) = \ln(W)$, so $\mathrm{E}[U(W)] = 0.5\ln(500) + 0.5\ln(600) \approx 6.3058$.

Certainty-equivalent wealth CE is given by

$$U(CE) = \mathrm{E}[U(W)] \quad \Leftrightarrow \quad \ln(CE) = 6.3058 \quad \Leftrightarrow \quad CE = e^{6.3058} = 547.74\,.$$

So $\pi_M = \mathrm{E}[W] - CE = 550 - 547.74 = 2.26$, so close to 2.50.

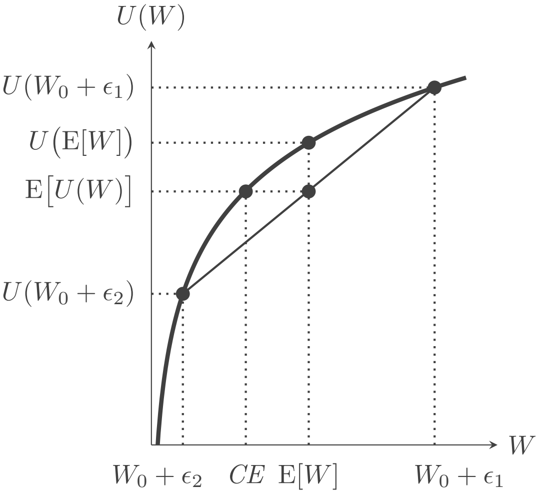
Finding the Markowitz risk premium geometrically



Remarks:

- Recall our figure for finding certainty equivalent wealth
- For risk-averse investors, $CE < \mathrm{E}[W]$ by definition

Finding the Markowitz risk premium geometrically



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- Recall our figure for finding certainty equivalent wealth
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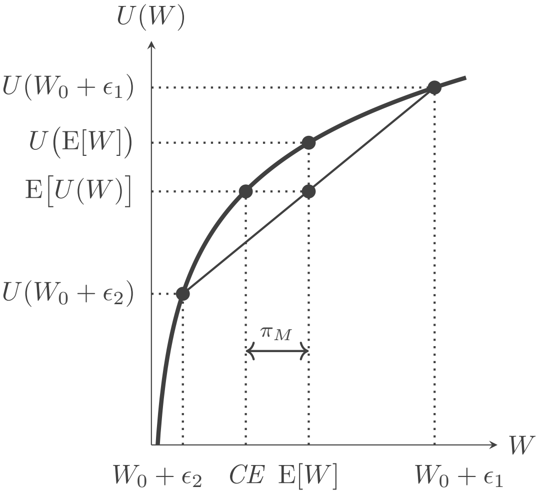
- The Markowitz risk premium is the difference $\mathrm{E}[W] - CE$

Finding the Markowitz risk premium geometrically



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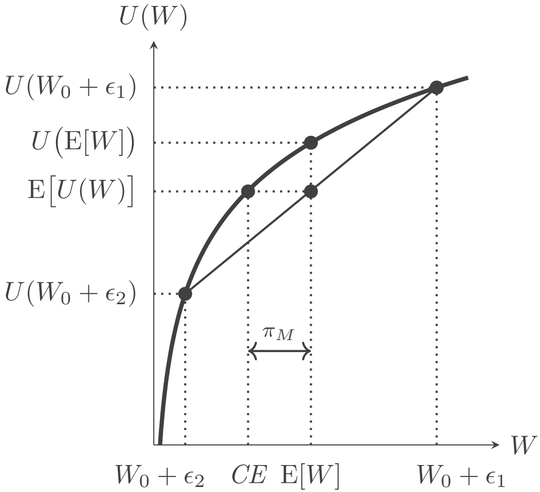
Finding the Markowitz risk premium geometrically



Remarks:

- Recall our figure for finding certainty equivalent wealth
- For risk-averse investors, $CE < \mathrm{E}[W]$ by definition
- The Markowitz risk premium is the difference $\mathrm{E}[W] - CE$
- The Markowitz risk premium is thus the difference shown at left
- For risk-averse investors, this risk premium is always positive

Quick summary: three key facts to remember about risk aversion

Fact 1: Risk aversion $\Rightarrow U\big(\mathrm{E}[W]\big) > \mathrm{E}\big[U(W)\big]$

Fact 2: Risk aversion $\Rightarrow CE < \mathrm{E}[W]$

Fact 3: Risk aversion $\Rightarrow \pi_M > 0$

Remarks:

- Check your understanding: can you explain each variable / operator / function?
- Can you draw these results in the space of wealth and utility?
- Can you derive these results for an example utility function, say $U(W) = -e^{-aW}$?

Lecture 05: Risk Aversion and Expected Utility II
Overview of Topics

Reading: Bodie et al. (2014, Ch 6)

The Pratt-Arrow approximation of the Markowitz risk premium

$$\pi_M \approx \frac{1}{2} \times ARA \times \text{Var}(\epsilon)$$

Remarks:

- Return to example with initial wealth W_0 and random lottery outcome ϵ
- Recall that $\text{Var}(\epsilon)$ measures the risk that outcome ϵ deviates from expectation
- Recall that ARA measures the investor's absolute level of aversion to risk
- Risk premium π_M is proportional to the level of risk aversion and the level of risk!
- Let's see how this is derived...

Deriving the Arrow-Pratt approximation (1/2)

Recall that the Markowitz risk premium satisfies

$$\mathrm{E}\big[U(W)\big] = U\big(\mathrm{E}[W] - \pi_M\big).$$

Let $W = W_0 + \epsilon$, with initial wealth W_0 and lottery outcome ϵ, where $\mathrm{E}[\epsilon] = 0$. Now,

$$\mathrm{E}\big[U(W_0 + \epsilon)\big] = U\big(\mathrm{E}[W_0 + \epsilon] - \pi_M\big) = U(W_0 - \pi_M).$$

Taking Taylor series expansions of both sides,

$$\mathrm{E}\Big[U(W_0) + \epsilon U'(W_0) + \frac{1}{2}\epsilon^2 U''(W_0) + \dots\Big] = U(W_0) - \pi_M U'(W_0) + \dots$$

Deriving the Arrow-Pratt approximation (2/2)

Using $\mathrm{E}[\epsilon] = 0$ and $\mathrm{Var}[\epsilon] = \mathrm{E}[\epsilon^2]$, and rearranging terms,

$$\pi_M \approx \frac{1}{2}\left(-\frac{U''(W_0)}{U'(W_0)}\right)\mathrm{Var}(\epsilon)$$

Recall that $ARA = -\frac{U''(W_0)}{U'(W_0)}$ is the absolute risk aversion of the investor, so

$$\pi_M \approx \frac{1}{2} \times ARA \times \mathrm{Var}(\epsilon).$$

Remarks:

- Given $U'(W_0) > 0$ and $U''(W_0) < 0$, the Pratt-Arrow risk premium is positive
- You should attempt to follow the derivation above, but it is not examinable

Lecture 05: Risk Aversion and Expected Utility II

Revision Checklist

- ☐ Certainty equivalent
- ☐ Markowitz risk premium
- ☐ Arrow-Pratt Approximation

Reading: Bodie et al. (2014, Ch 6)

Intermediate Finance
Table of Contents

Lecture 06: Optimal Portfolio Selection I
Overview of Topics

Reading: Hillier et al. (2016, Ch 10), Bodie et al. (2014, Ch 6, 7, & 8)

Portfolio of N Assets

Definition 8

Portfolio	A collection of assets held by an investor. Once established, the portfolio can be updated by buying or selling assets.

Remarks:

- We consider a portfolio of N assets with returns $R_1, R_2, R_3, \ldots, R_N$
- We want to know: what are the expectation and variance of portfolio returns?

First, let's see how realized portfolio returns are computed

$$R_p = \sum_{i=1}^{N} w_i R_i \quad \text{where} \quad \sum_{i=1}^{N} w_i = 1$$

Remarks:

- R_p return on portfolio
- R_i return on asset i, where $i \in \{1, 2, \ldots, N\}$
- w_i portfolio weight on asset i (non-random because chosen by investor)

- Realized portfolio returns are a weighted average of realized asset returns
- The portfolio weight equals the fraction of wealth invested in asset i
- The portfolio could contain many different assets and asset classes

Lecture 06: Optimal Portfolio Selection I
Overview of Topics

Reading: Hillier et al. (2016, Ch 10), Bodie et al. (2014, Ch 6, 7, & 8)

Expectation of Portfolio Returns

$$\mu_p = \mathrm{E}[R_p] = \mathrm{E}\left[\sum_{i=1}^{N} w_i R_i\right] = \sum_{i=1}^{N} w_i \mathrm{E}[R_i] = \sum_{i=1}^{N} w_i \mu_i$$

Notation and remarks:

$\mathrm{E}[R_p]$	Expected return on portfolio, also denoted μ_p
$\mathrm{E}[R_i]$	Expected return on asset i, also denoted μ_i, where $i \in \{1, 2, \ldots, N\}$

- Expected portfolio return is a weighted average of expected asset returns
- Note that the weights are non-random—they are chosen by the investor!

Covariance of returns on assets i and j

$$\sigma_{ij} = \mathrm{Cov}\big(R_i, R_j\big) = \mathrm{E}\Big[\Big(R_i - \mathrm{E}[R_i]\Big)\Big(R_j - \mathrm{E}[R_j]\Big)\Big] = \mathrm{E}\Big[(R_i - \mu_i)(R_j - \mu_j)\Big]$$

Notation and remarks:

$\mathrm{Cov}\big(R_i, R_j\big)$ Covariance of returns on assets i and j, also denoted σ_{ij}, $i, j \in \{1, 2, \ldots, N\}$

- Covariance measures the tendency of random variables to move together
- More asset return comovement implies higher portfolio return variance
- The correlation between returns is given by $\rho_{ij} = \sigma_{ij}/\sigma_i\sigma_j \in [-1, +1]$

Variance of Portfolio Returns

$$\sigma_p^2 = \mathrm{Var}(R_p) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \mathrm{Cov}(R_i, R_j)$$

Notation:

$\mathrm{Var}(R_p)$	Variance of return on portfolio, also denoted σ_p^2
$\mathrm{Var}(R_i)$	Variance of return on asset i, also denoted σ_i^2, where $i \in \{1, 2, \ldots, N\}$

- Portfolio variance equals the weighted sum of asset variances and covariances

Variance of Portfolio Returns

$$\sigma_p^2 = \mathrm{Var}(R_p) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \mathrm{Cov}(R_i, R_j)$$
$$= \sum_{i=1}^{N} w_i^2 \mathrm{Var}(R_i) + \sum_{i=1}^{N}\sum_{j\neq i}^{N} w_i w_j \mathrm{Cov}(R_i, R_j)$$

Notation:

$\mathrm{Var}(R_p)$	Variance of return on portfolio, also denoted σ_p^2
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Variance of Portfolio Returns

$$\begin{aligned}\sigma_p^2 = \mathrm{Var}(R_p) &= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \mathrm{Cov}(R_i, R_j)\\ &= \sum_{i=1}^{N} w_i^2 \mathrm{Var}(R_i) + \sum_{i=1}^{N}\sum_{j\neq i}^{N} w_i w_j \mathrm{Cov}(R_i, R_j)\\ &= \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N}\sum_{j\neq i}^{N} w_i w_j \sigma_{ij}\end{aligned}$$

Notation:

$\mathrm{Var}(R_p)$ Variance of return on portfolio, also denoted σ_p^2

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Lecture 06: Optimal Portfolio Selection I
Overview of Topics

6.1. Portfolios with N assets

6.2. Expectation and variance of portfolio returns

6.3. Naive diversification

6.4. Two-Assets with correlated returns

Reading: Hillier et al. (2016, Ch 10), Bodie et al. (2014, Ch 6, 7, & 8)

Question 9

You've heard that diversification can lower your risk. You decide to diversify by holding an equal-weighted portfolio of randomly-chosen stocks. Roughly how many stocks will you need, before you consider yourself "well-diversified"?

A. 3 or 4 stocks should be enough

B. 30 or 40 stocks should be enough

C. 300 or 400 stocks should be enough

Question 9

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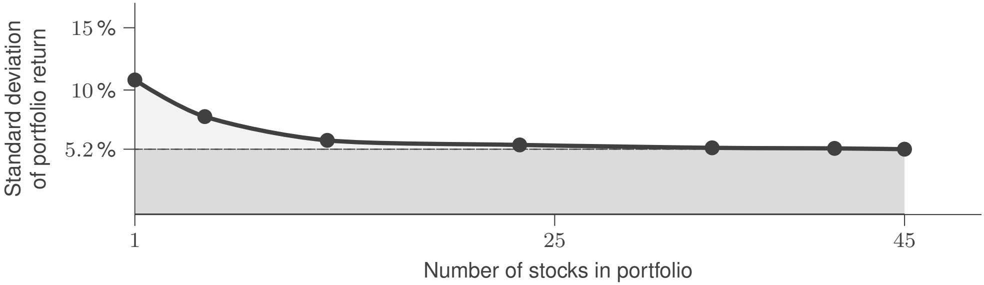
Naive diversification: larger portfolios are less volatile



Remarks:

- Median return on 1,000 random portfolios of sizes: 1, 5, 12, 23, 34, 41, 45 stocks
- Light-gray shading is diversifiable idiosyncratic risk, dark gray is systematic risk
- Standard deviations are computed from annualized monthly returns, 1970-2015
- Computed for equal-weighted portfolio returns from a CRSP balanced panel

How does naive diversification work? A simplified example

$$\text{Recall} \quad \sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j \neq i}^{N} w_i w_j \sigma_{ij}$$

$$\begin{aligned} \text{Now} \quad \sigma_p^2 &= \frac{N}{N^2} v + \frac{N(N-1)}{N^2} c \\ &= \frac{1}{N} v + \left(1 - \frac{1}{N}\right) c = \frac{1}{N}(v - c) + c \end{aligned}$$

Remarks:

- To simplify, assume equal variances $\sigma_i^2 = v$, covariances $\sigma_{ij} = c$, weights $w_i = 1/N$
- The equation for the variance of portfolio returns can now be simplified greatly
- As $N \to \infty$, idiosyncratic part $(1/N)(v - c) \to 0$, and only systematic part c remains

Naive versus efficient diversification

- In naive diversification, we equally weighted a random selection of assets for our portfolios
- Diversification worked slowly because we didn’t exploit information on asset covariances
- Can we choose weights to diversify efficiently rather than diversifying randomly?
- Yes! Next lecture we find the minimum-variance portfolio by combining two assets efficiently

Lecture 06: Optimal Portfolio Selection I
Overview of Topics

Reading: Hillier et al. (2016, Ch 10), Bodie et al. (2014, Ch 6, 7, & 8)

We study three cases of two-asset portfolios

Case 1: perfect positively correlated asset returns ($\rho_{AB} = 1$)

Case 2: perfect negatively correlated asset returns ($\rho_{AB} = -1$)

Case 3: imperfectly correlated asset returns ($-1 < \rho_{AB} < 1$)

Remarks:

- We assume an investor can choose between two risky assets A and B
- We constrain portfolio weights w_A and w_B to sum to one: $w_A + w_B = 1$
- Special case: for two assets, $\sigma_P^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_{AB}$
- Special case: for two assets, $\mu_P = w_A\mu_A + w_B\mu_B$

Case 1: perfect positive correlation, $\rho_{AB} = 1$

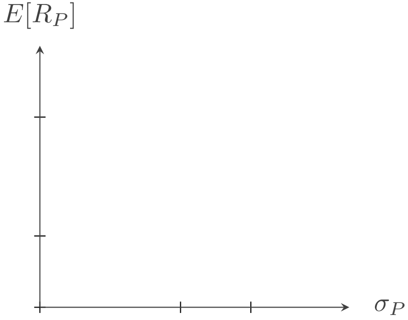
Because $\rho_{AB} = \sigma_{AB}/(\sigma_A \sigma_B) = 1$,

$$\begin{aligned} \sigma_P^2 &= w_A^2 \sigma_A^2 + 2 w_A w_B \sigma_A \sigma_B + w_B^2 \sigma_B^2 \\ &= (w_A \sigma_A + w_B \sigma_B)^2 \\ \Leftrightarrow \quad \sigma_P &= w_A \sigma_A + w_B \sigma_B \end{aligned}$$

Remarks:

- In this case, portfolio risk is the weighted average of asset A and B risks
- In this case, returns on assets A and B always move in the same direction
- These assets lead to no diversification, because no offsetting movements occur

Case 1: perfect positive correlation, $\rho_{AB} = 1$



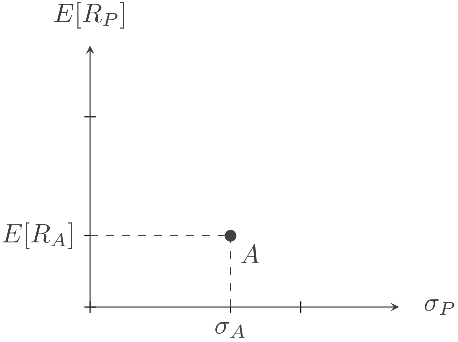
With $w_B = 1 - w_A$, σ_P and μ_P are

$$\mu_P = w_A(\mu_A - \mu_B) + \mu_B$$
$$\sigma_P = w_A(\sigma_A - \sigma_B) + \sigma_B.$$

Combine to get linear relationship:

$$\mu_P = \frac{\sigma_A \mu_B - \sigma_B \mu_A}{\sigma_A - \sigma_B} + \frac{\mu_A - \mu_B}{\sigma_A - \sigma_B} \sigma_P.$$

Case 1: perfect positive correlation, $\rho_{AB} = 1$



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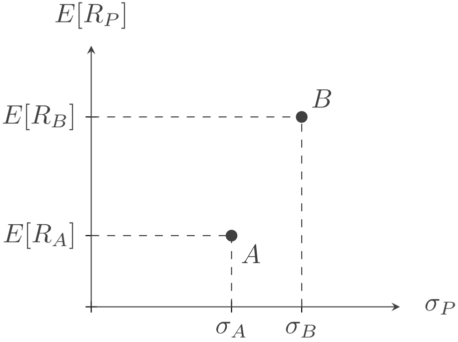

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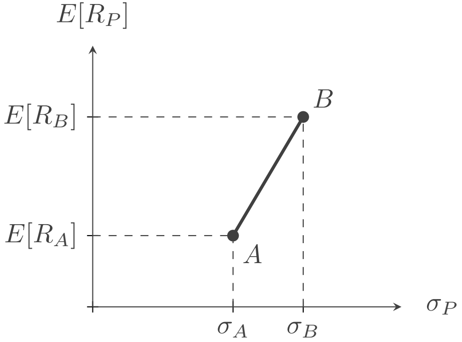

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$$\mu_P = \frac{\sigma_A \mu_B - \sigma_B \mu_A}{\sigma_A - \sigma_B} + \frac{\mu_A - \mu_B}{\sigma_A - \sigma_B}\sigma_P.$$

Case 2: perfect negative correlation, $\rho_{AB} = -1$

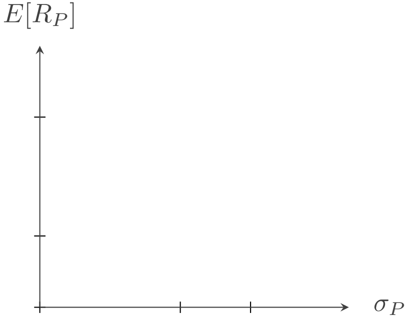
Because $\rho_{AB} = \sigma_{AB}/(\sigma_A \sigma_B) = -1$,

$$\begin{aligned} \sigma_P^2 &= w_A^2 \sigma_A^2 - 2 w_A w_B \sigma_A \sigma_B + w_B^2 \sigma_B^2 \\ &= (w_A \sigma_A - w_B \sigma_B)^2 \\ \Leftrightarrow \sigma_P &= \pm (w_A \sigma_A - w_B \sigma_B). \end{aligned}$$

Remarks:

- In this case, portfolio risk is the difference between weighted risks of A and B
- Since σ_P is positive, $w_A \sigma_A > w_B \sigma_B \Rightarrow$ "$+$" case and $w_A \sigma_A < w_B \sigma_B \Rightarrow$ "$-$" case
- These assets lead to perfect diversification, because their movements offset

Case 2: perfect negative correlation, $\rho_{AB} = -1$



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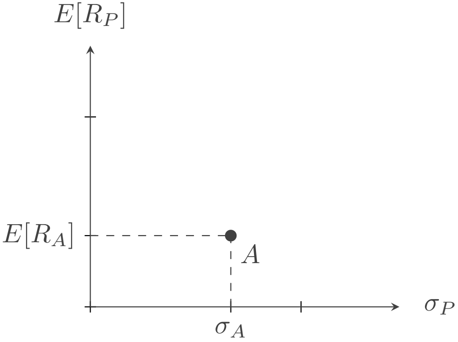

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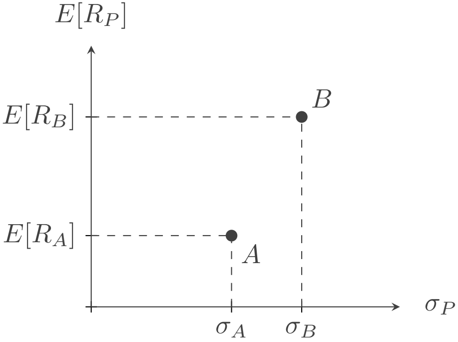

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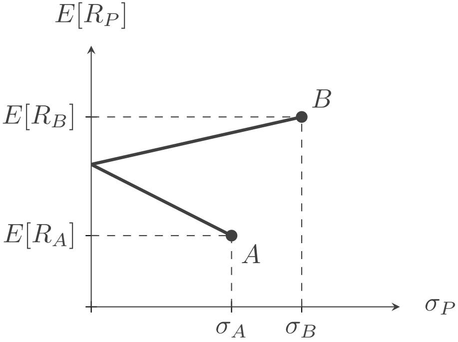

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Case 2: perfect negative correlation, $\rho_{AB} = -1$

Question 10

Two risky assets have returns with standard deviations σ_A and σ_B. The returns are perfectly negatively correlated. What weights w_A and w_B will reduce to zero the variance of returns on an AB-portfolio?

A. $w_A = \sigma_B/(\sigma_A + \sigma_B)$ and $w_B = \sigma_A/(\sigma_A + \sigma_B)$

B. $w_A = \sigma_A/(\sigma_A + \sigma_B)$ and $w_B = \sigma_B/(\sigma_A + \sigma_B)$

Remarks:

- Two assets with perfectly negatively correlated returns can form a riskless portfolio
- To find the required weights, simply set the expression for portfolio variance to zero

Case 2: perfect negative correlation, $\rho_{AB} = -1$

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Remarks:

- Two assets with perfectly negatively correlated returns can form a riskless portfolio
- To find the required weights, simply set the expression for portfolio variance to zero

Case 2: perfect negative correlation, $\rho_{AB} = -1$

Solution 10

With perfect negative correlation, portfolio risk can be reduced to zero:

$$\sigma_P = \pm(w_A\sigma_A - w_B\sigma_B) = 0 \quad \Leftrightarrow \quad w_A\sigma_A - (1 - w_A)\sigma_B = 0\,.$$

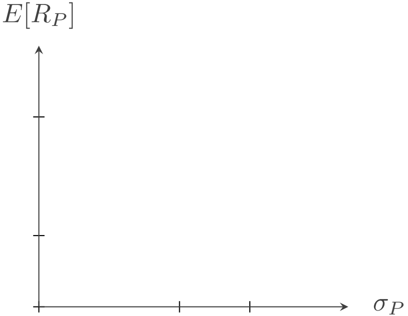
Rearranging the expression on the right,

$$w_A = \frac{\sigma_B}{\sigma_A + \sigma_B}\,, \qquad w_B = 1 - w_A = \frac{\sigma_A}{\sigma_A + \sigma_B}\,.$$

The expected return on the perfect hedge portfolio is then:

$$\mu_P = \frac{\sigma_B}{\sigma_A + \sigma_B}\mu_A + \frac{\sigma_A}{\sigma_A + \sigma_B}\mu_B\,.$$

Case 3: Imperfect correlation, $-1 < \rho_{AB} < 1$



Remarks:

- The general case of imperfectly-correlated returns is most relevant empirically
- In the next lecture, we will study this case carefully, and construct optimal portfolios
- Even with positive ρ_{AB}, portfolios offer higher expected returns for given variance
- Even with positive ρ_{AB}, portfolios offer lower variance for given expected returns

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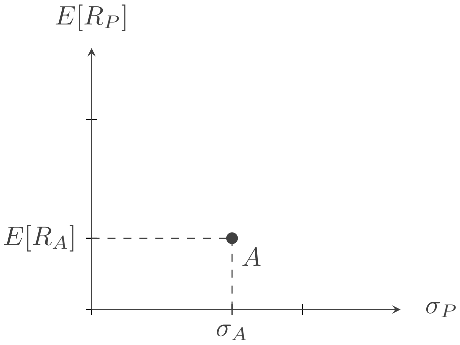

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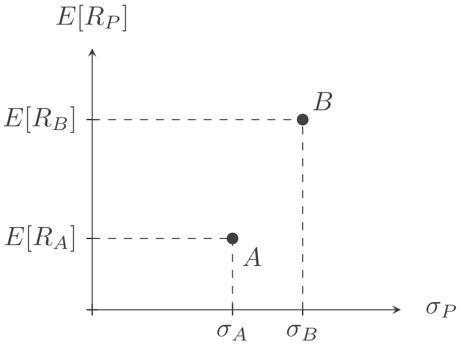

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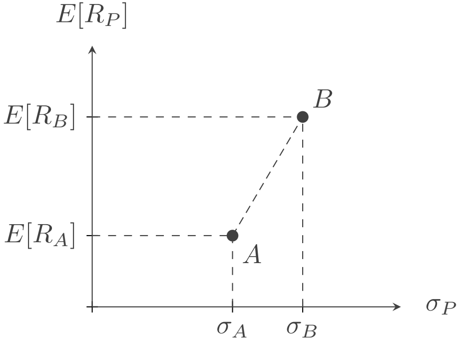

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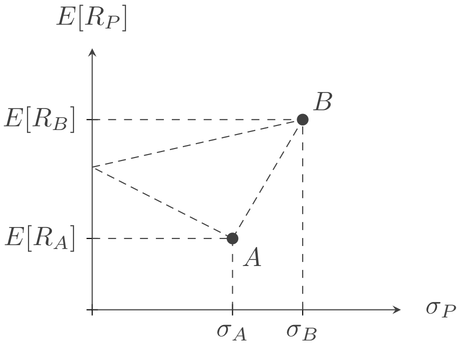

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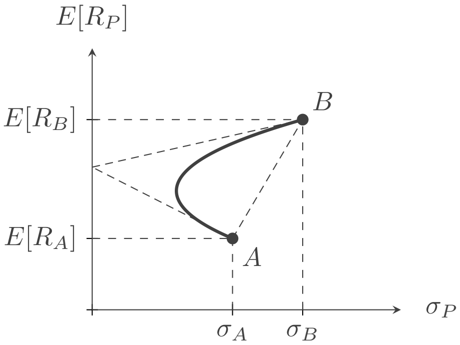

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Lecture 06: Optimal Portfolio Selection I

Revision Checklist

- ☐ Portfolios with N assets
- ☐ Expectation and variance of portfolio returns
- ☐ Naive diversification
- ☐ Two-Assets with correlated returns

Reading: Hillier et al. (2016, Ch 10), Bodie et al. (2014, Ch 6, 7, & 8)

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Lecture 07: Optimal Portfolio Selection II
Overview of Topics

7.1. Minimum variance portfolio

7.2. Capital allocation line

7.3. Finding the tangency portfolio

7.4. Lending and borrowing portfolios

Reading: Hillier et al. (2016, Ch 10), Bodie et al. (2014, Ch 6, 7, & 8), Solnik (1974, optional), Elton et al. (2011, optional)

Recall the three cases of two-asset portfolios

Case 1: perfect positively correlated asset returns ($\rho_{AB} = 1$)

Case 2: perfect negatively correlated asset returns ($\rho_{AB} = -1$)

Case 3: imperfectly correlated asset returns ($-1 < \rho_{AB} < 1$)

Remarks:

- We now focus on case 3: imperfectly correlated returns on assets A and B
- Can we choose weights that minimize the variance of returns on an AB-portfolio?
- This would be efficient diversification, not naive equal-weighted diversification

Finding the Variance-Minimizing Portfolio Weights

$$\min_{\text{w.r.t. } w_A, w_B} \quad \sigma_P^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_{AB}$$

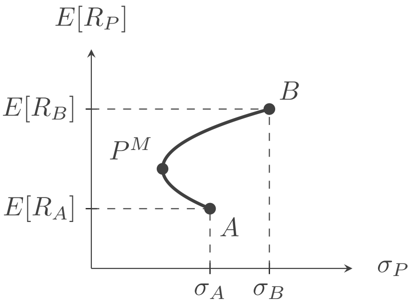
$$\text{s.t.} \quad 1 = w_A + w_B$$

Notation and remarks:

$\min$ Minimization operator, used to minimize functions by choice of argument

- To solve the minimization, first replace w_B with $1 - w_A$ in the expression for σ_P^2
- Then derive the first-order condition by setting derivative of σ_P^2 w.r.t. w_A to zero
- This gives you an expression to solve for w_A i.t.o. knowns σ_A, σ_B, and σ_{AB}
- The resulting value for w_A will be the variance-minimizing weight on asset A
- Abbreviations: "w.r.t."="with respect to", "s.t."="subject to", "i.t.o."="in terms of"

Minimum Variance Portfolio



The minimization problem above implies

$$w_A^M = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B} .$$

To find the variance-minimizing w_B^M, use

$$w_B^M = 1 - w_A^M .$$

Remarks:

- The point P^M in the diagram above represents the minimum variance portfolio
- The portfolio P^M is efficiently diversified in the sense that variance is minimized
- But is P^M the portfolio that risk-averse investors should hold? Not at all!

Lecture 07: Optimal Portfolio Selection II
Overview of Topics

Reading: Hillier et al. (2016, Ch 10), Bodie et al. (2014, Ch 6, 7, & 8), Solnik (1974, optional), Elton et al. (2011, optional)

The Sharpe Ratio: Reward over Risk

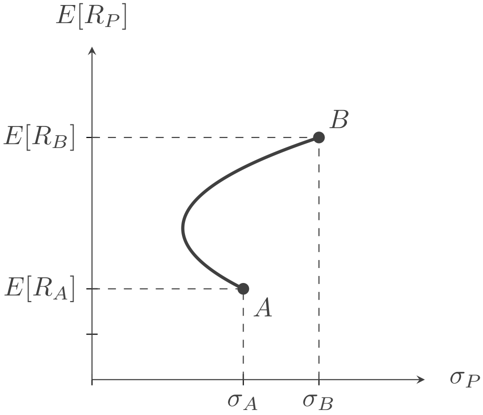
$$\text{Sharpe Ratio:} \quad SR = \frac{\mu_P - R_f}{\sigma_P}$$

Notation and remarks:

R_f	Risk-free rate, think of the interest rate on short-term treasuries or similar
$\mu_P - R_f$	Expected excess return, i.e. the return you earn above the risk-free rate

- The Sharpe Ratio is the ratio of reward to risk for an asset or portfolio
- The Sharpe Ratio equals the expected excess return per unit of std dev of return
- By definition, the risk-free rate R_f is non-random, so $\mathrm{E}[R_f] = R_f$

Capital Allocation Line shows portfolios of riskless and risky assets



Create a new portfolio by combining a risky AB-portfolio with a riskless asset.

Denote the new portfolio expected return $\mu_{P'}$ and standard deviation $\sigma_{P'}$.

Using $w_P + w_f = 1$ and $\sigma_{Pf} = \sigma_f = 0$,

$$\mu_{P'} = w_P(\mu_P - \mu_f) + \mu_f$$
$$\sigma_{P'} = w_P \sigma_P.$$

Combining, we see that the slope of the CAL equals the Sharpe Ratio:

$$\mu_{P'} = \frac{\mu_P - \mu_f}{\sigma_P} \sigma_{P'} + \mu_f.$$

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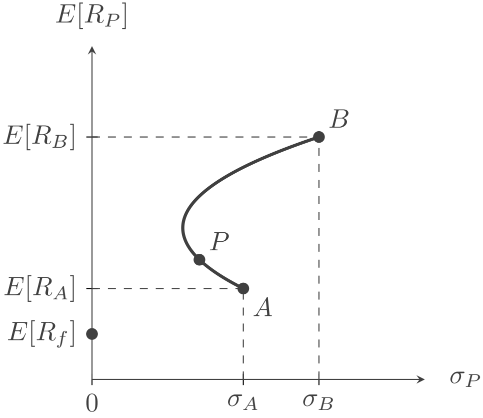

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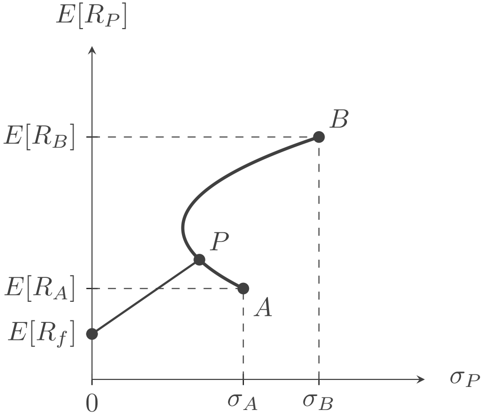

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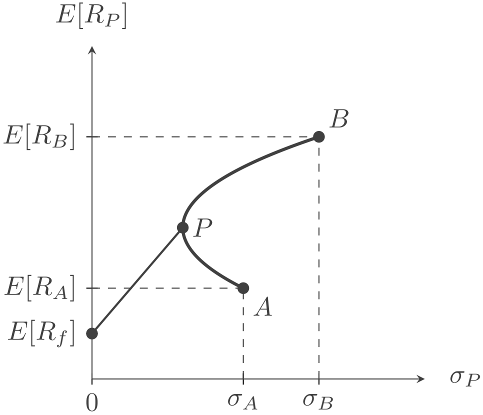

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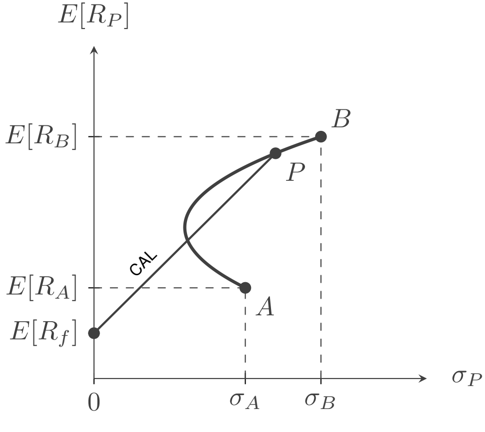

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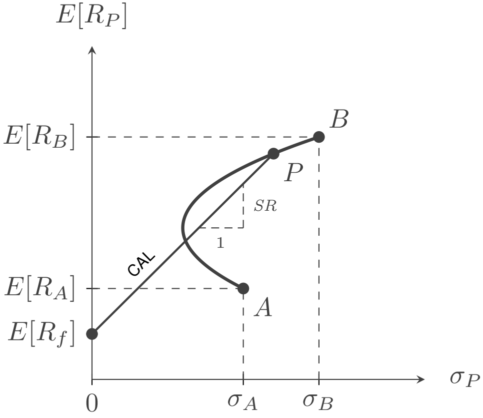

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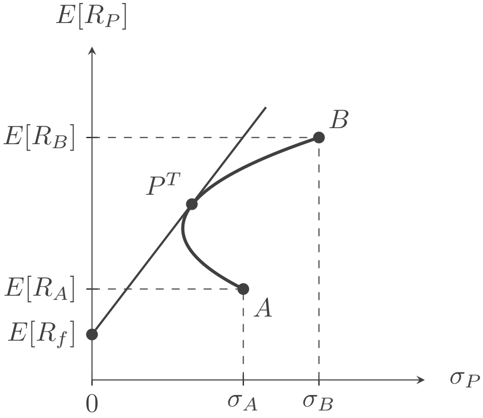
The tangency portfolio maximizes the Sharpe Ratio



The CAL combines the riskless asset with any risky AB-portfolio

Of all possible risky AB-portfolios, which best combines with the riskless asset?

Tangency portfolio P^T maximizes the slope of the CAL and thus the Sharpe Ratio

Portfolio P^T + riskless asset therefore yields the highest reward-to-risk ratio

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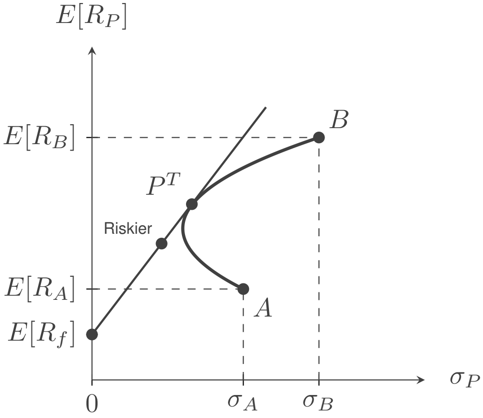

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Investors then apply more or less weight to riskless asset, depending on risk tolerance

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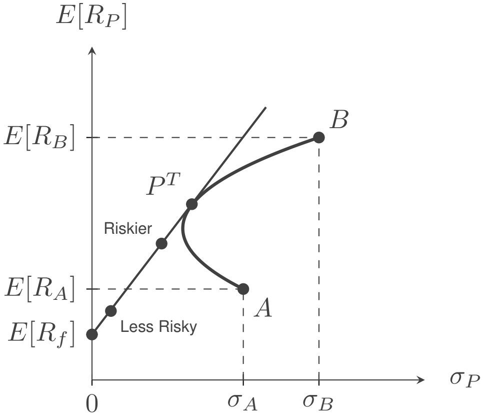

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Lecture 07: Optimal Portfolio Selection II
Overview of Topics

Reading: Hillier et al. (2016, Ch 10), Bodie et al. (2014, Ch 6, 7, & 8), Solnik (1974, optional), Elton et al. (2011, optional)

Finding the tangency portfolio: specifying the problem

$$\max_{\text{w.r.t. } w_A, w_B} \quad \frac{\mu_P - R_f}{\sigma_P}$$

$$\text{s.t.} \quad 1 = w_A + w_B$$

Remarks:

- The tangency portfolio P^T is the optimal risky portfolio that all investors should hold
- How to weight the riskless asset and risky portfolio to obtain the tangency portfolio P^T?
- We choose w_A and w_B to maximize the Sharpe Ratio subject to $1 = w_A + w_B$

Finding the tangency portfolio: first-order conditions for two assets

$$\begin{bmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{bmatrix} \begin{bmatrix} w_A \\ w_B \end{bmatrix} = \lambda \begin{bmatrix} \mu_A - R_f \\ \mu_B - R_f \end{bmatrix}$$

Remarks:

- The weights w_A and w_B on the tangency portfolio satisfy the above matrix equation
- This matrix equation is a system of two simultaneous equations in λ and the weights
- Together with constraint $w_A + w_B = 1$, there are three equations and three unknowns
- The system can be solved for the weights that yield the tangency portfolio

Finding the tangency portfolio: an optimality condition

$$\frac{\mu_A - R_f}{\partial \sigma_P^2 / \partial w_A} = \frac{\mu_B - R_f}{\partial \sigma_P^2 / \partial w_B}$$

Remarks:

- An optimality condition gives us useful intuition for choosing optimal weights
- Optimal weights equalize reward over marginal increase in portfolio risk across assets
- We can derive this optimality condition using the first-order conditions from above

Lecture 07: Optimal Portfolio Selection II
Overview of Topics

Reading: Hillier et al. (2016, Ch 10), Bodie et al. (2014, Ch 6, 7, & 8), Solnik (1974, optional), Elton et al. (2011, optional)

Combining the riskless asset and the tangency portfolio

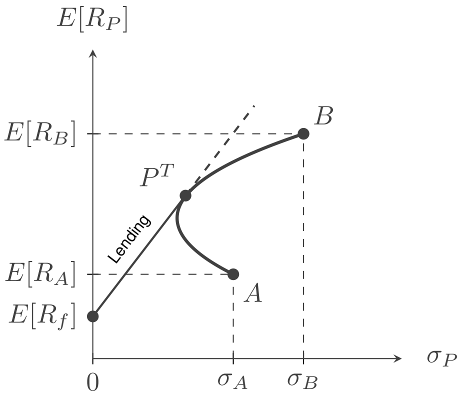
More risk-averse: optimal allocation combines risk-free lending with portfolio P^T

Less risk-averse: optimal allocation combines risk-free borrowing with portfolio P^T

Remarks:

- We have solved for the tangency portfolio by choosing weights w_A and w_B
- Next we must allocate wealth between tangency portfolio P^T and the risk-free asset
- This decision depends on individual risk preferences, and may involve leverage

Lending Portfolios



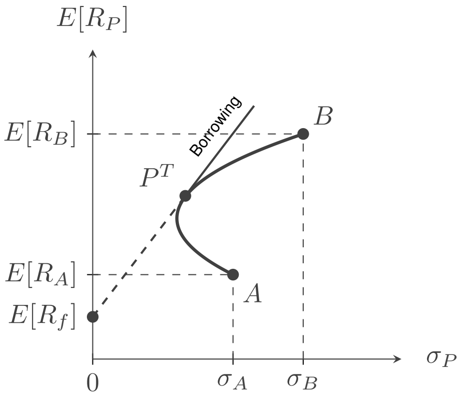
More risk averse investors may want less risk than portfolio P^T offers

These investors will allocate some wealth toward the risk-free asset

Thus, they will lend funds to others and earn the risk-free rate

Portfolios on the solid blue portion of the CAL are lending portfolios

Borrowing Portfolios



Less risk averse investors may want more risk than portfolio P^T offers

These investors will use leverage to invest more funds in the portfolio P^T

Thus, they will borrow funds from others and pay the risk-free rate

Portfolios on the solid blue portion of the CAL are borrowing portfolios

Lecture 07: Optimal Portfolio Selection II

Revision Checklist

- ☐ Minimum variance portfolio
- ☐ Capital allocation line
- ☐ Finding the tangency portfolio
- ☐ Lending and borrowing portfolios

Reading: Hillier et al. (2016, Ch 10), Bodie et al. (2014, Ch 6, 7, & 8), Solnik (1974, optional), Elton et al. (2011, optional)

Intermediate Finance
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Lecture 08: Capital Asset Pricing Model I
Overview of Topics

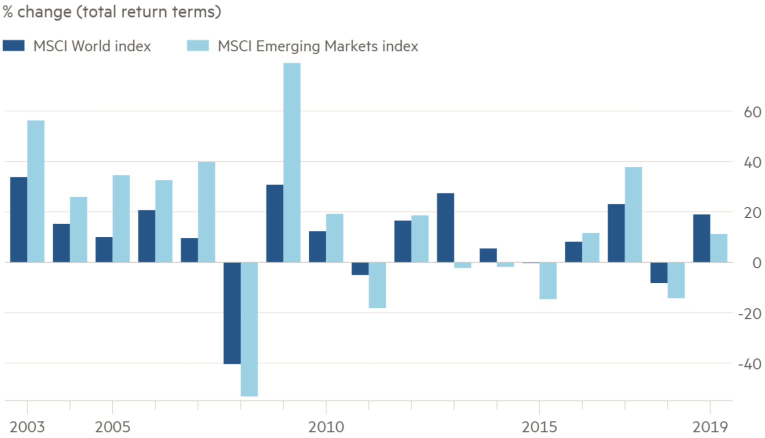
8.1. Comovement with portfolio returns

8.2. Derivation of beta

8.3. Efficient frontier with N assets

Reading: Hillier et al. (2016, Ch 10 & 12), Bodie et al. (2014, Ch 9), Fama and French (1992)

MSCI Emerging moves with but more extremely than MSCI World



Source: *Emerging Market Equities Flunk Test of High "Beta" Status*, Financial Times, August 2019 (link)

Note: MSCI World excludes Emerging Markets, despite what the name suggests. (link)

Beta tells us what an asset contributions to portfolio variance

$$\beta_e = \frac{\mathrm{Cov}(R_e, R_p)}{\mathrm{Var}(R_p)}$$

Notation and remarks:

- R_e Return on the MSCI Emerging Market Index
- R_p Return on your MSCI World and Emerging Markets portfolio
- You're risk averse, so you dislike variance of returns on your portfolio
- You hold MSCI World: large and mid cap stocks from developed markets
- By adding MSCI Emerging, how will your portfolio variance change?
- Your portfolio variance will rise if $\beta_e > 1$ and fall if $\beta_e < 1$. Let's see why!

Lecture 08: Capital Asset Pricing Model I
Overview of Topics

Reading: Hillier et al. (2016, Ch 10 & 12), Bodie et al. (2014, Ch 9), Fama and French (1992)

Deriving beta step 1: write variance as sum of portfolio covariances

$$
\begin{aligned}
\operatorname{Var}(R_P) &= \mathrm{E}\Big[\big(R_P - \mathrm{E}[R_P]\big)^2\Big] \\
&= \sum_{i=1}^{N} w_i \mathrm{E}\Big[\big(R_i - \mathrm{E}[R_i]\big)\big(R_p - \mathrm{E}[R_P]\big)\Big] \\
&= \sum_{i=1}^{N} w_i \operatorname{Cov}(R_i, R_P)
\end{aligned}
$$

Remarks:

- Start with the variance of returns on a portfolio P of N risky assets i
- Portfolio variance is a weighted sum of covariances b/w assets i and portfolio P
- The right-hand side shows each asset i's contribution to portfolio variance
- This starts a simple, intuitive derivation; other more formal approaches exist...

Deriving beta step 2: normalize covariances by portfolio variance

$$1 = \sum_{i=1}^{N} w_i \frac{\text{Cov}(R_i, R_P)}{\text{Var}(R_P)} = \sum_{i=1}^{N} w_i \beta_i = \beta_P$$

Remarks:

- Beta offers a relative measure of comovement between asset and portfolio returns
- By definition of beta, $\beta_P = 1$, and the useful expression $\beta_P = \sum_{i=1}^{N} w_i \beta_i$ holds
- Assets with $\beta_i < 1$ tend to lower portfolio variance, assets with $\beta_i > 1$ tend to raise it
- Hence, risk-averse investors demand higher returns on assets with higher beta!

Numerical example: computing portfolio betas

Question 11

You invest 75% of your wealth in the MSCI World Index, with a variance of returns equal to 0.02, and 25% of your wealth in the MSCI Emerging Markets Index, with a variance of returns equal to 0.04. Returns on these indices have a covariance of 0.015. Which index has a higher portfolio beta?

A. MSCI World Index

B. MSCI Emerging Market Index

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A. MSCI World Index

B. **MSCI Emerging Market Index**

Numerical example: computing portfolio betas

Solution 11

With World, Emerging, and portfolio subscripts w, e, and p, respectively, we have

$$\begin{aligned} \sigma_{pw} &= \sigma_w^2 w_w + \sigma_{we} w_e \\ \sigma_{pe} &= \sigma_e^2 w_e + \sigma_{we} w_w \\ \sigma_p^2 &= \sigma_w^2 w_w^2 + \sigma_e^2 w_e^2 + 2 w_w w_e \sigma_{we}. \end{aligned}$$

With above equations and the following values, we can compute β_w and β_e:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_w^2 & \sigma_{we} \\ \sigma_{ew} & \sigma_e^2 \end{bmatrix} = \begin{bmatrix} 0.02 & 0.015 \\ 0.015 & 0.04 \end{bmatrix}, \qquad w_w = 0.75, \quad w_e = 0.25.$$

Next, use this information to compute variances and covariance...

Numerical example: computing portfolio betas

Solution 11

First compute portfolio variance and covariances as

$$\sigma_{pw} = 0.02 \cdot 0.75 + 0.015 \cdot 0.25 = 0.01875$$
$$\sigma_{pe} = 0.04 \cdot 0.25 + 0.015 \cdot 0.75 = 0.02125$$
$$\sigma_p^2 = 0.02 \cdot 0.75^2 + 0.04 \cdot 0.25^2 + 2 \cdot 0.75 \cdot 0.25 \cdot 0.015 = 0.019375.$$

Now calculating β_e and β_w and comparing, we have

$$\beta_w = \frac{\sigma_{pw}}{\sigma_p^2} = 0.9677 < 1.3548 = \frac{\sigma_{pe}}{\sigma_p^2} = \beta_e.$$

Lecture 08: Capital Asset Pricing Model I
Overview of Topics

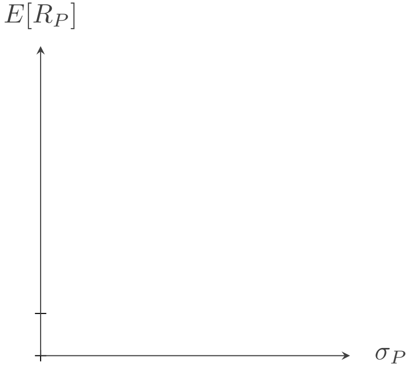
8.1. Comovement with portfolio returns

8.2. Derivation of beta

8.3. Efficient frontier with N assets

Reading: Hillier et al. (2016, Ch 10 & 12), Bodie et al. (2014, Ch 9), Fama and French (1992)

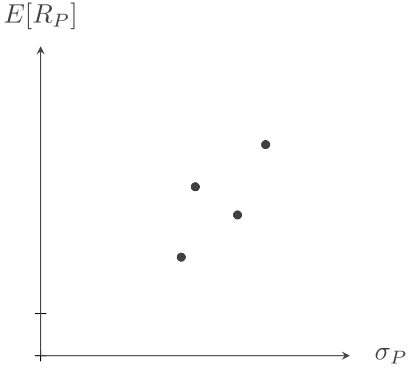
The efficient frontier dominates all other portfolios



Remarks:

- What does the feasible set of portfolios look like with more than two assets?
- A bullet-shaped set of possible portfolios forms as assets are combined
- Any portfolio inside the bullet is feasible, but many are strictly dominated
- The set of mean-variance efficient portfolios forms an efficient frontier.
- The portfolio P^T tangent to the frontier dominates all other risky portfolios

The efficient frontier dominates all other portfolios



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- What does the feasible set of portfolios look like with more than two assets?
- A bullet-shaped set of possible portfolios forms as assets are combined
- Any portfolio inside the bullet is feasible, but many are strictly dominated
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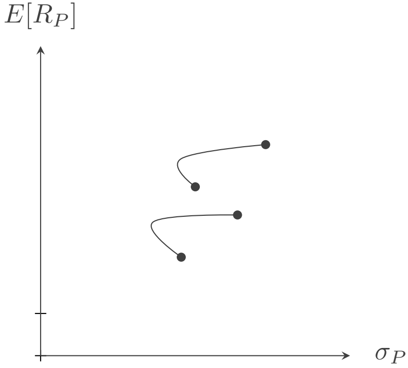

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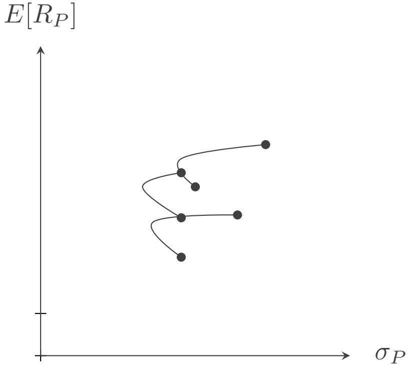

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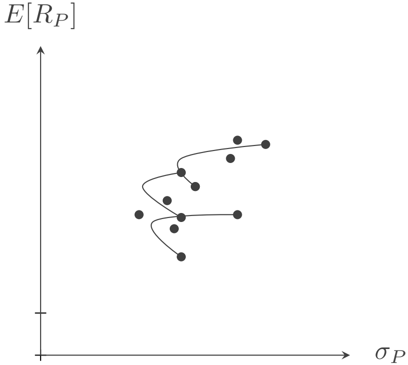

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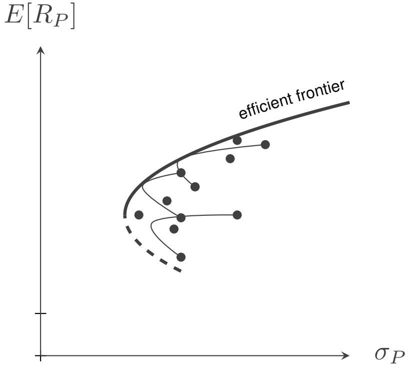

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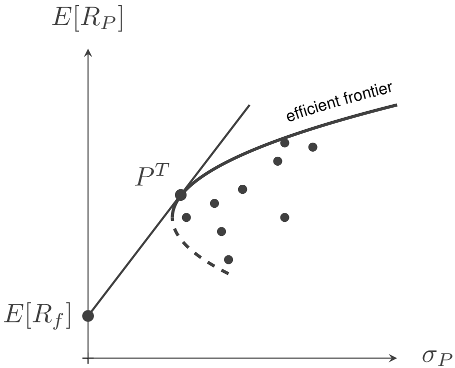

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Finding the tangency portfolio: first-order conditions for N assets

$$\begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22}^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN}^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = \lambda \begin{bmatrix} \mu_1 - R_f \\ \mu_2 - R_f \\ \vdots \\ \mu_N - R_f \end{bmatrix}$$

Remarks:

- Weights $w_1, w_2 \ldots w_N$ on the tangency portfolio satisfy the above matrix equation
- This matrix equation represents a system of N simultaneous equations in the weights
- Together with the constraint $\sum_{i=1}^{N} w_i = 1$, there are $N + 1$ equations and unknowns
- This system can be solved for the weights that yield the tangency portfolio

Lecture 08: Capital Asset Pricing Model I

Revision Checklist

- ☐ Comovement with portfolio returns
- ☐ Derivation of beta
- ☐ Efficient frontier with N assets

Reading: Hillier et al. (2016, Ch 10 & 12), Bodie et al. (2014, Ch 9), Fama and French (1992)

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Lecture 09: Capital Asset Pricing Model II
Overview of Topics

9.1. Assumptions of the capital asset pricing model
9.2. Deriving the capital asset pricing model
9.3. Securities market line
9.4. Empirical failures of the capital asset pricing model

Reading: Hillier et al. (2016, Ch 10, & 12), Bodie et al. (2014, Ch 9)

CAPM Assumptions

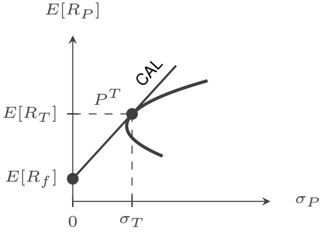
Individual behaviour

- Investors aim to achieve the optimal risk-return ratio
- Investors have single-period planning horizons
- Investors have homogeneous expectations

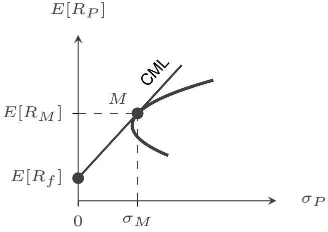
Market structure

- All assets are publicly held and tradable
- Investors can lend or borrow at the risk free rate
- All information is publicly available
- There are no taxes, transaction costs, or other frictions
- Short selling of risky assets is unrestricted

From the capital allocation line to the capital market line



- **CAL:** Investors identify a unique risky portfolio P^T to combine optimally with the risk-free asset
- Investors with different risk preferences will choose different (borrowing/lending) weights on the risk-free asset
- Any risky asset not in the tangency portfolio will not be held by any rational investor



- **CML:** Under CAPM assumptions, all investors combine riskless borrowing/lending with the tangency portfolio
- The tangency portfolio must therefore be the market portfolio M, and the optimal CAL becomes the Capital Market Line
- All investors will hold some combination of the market portfolio and the risk-free asset

Lecture 09: Capital Asset Pricing Model II
Overview of Topics

Reading: Hillier et al. (2016, Ch 10, & 12), Bodie et al. (2014, Ch 9)

CAPM Derivation: the optimality condition for tangency weights (1/3)

$$\frac{\mathrm{E}[R_i] - R_f}{\partial \sigma_M^2 / \partial w_i} = \frac{\mathrm{E}[R_j] - R_f}{\partial \sigma_M^2 / \partial w_j} \qquad \Rightarrow \qquad \frac{\mathrm{E}[R_i] - R_f}{\sigma_{iM}} = \frac{\mathrm{E}[R_M] - R_f}{\sigma_M^2}$$

Remarks:

- Recall the optimality condition above left from Sharpe Ratio maximization in lecture 7
- The optimality condition holds for any portfolio P, so choose $P = M$, i.e. the market
- From definition of portfolio variance, notice that $\partial \sigma_M^2 / \partial w_j \approx \sigma_{jM}$, and choose $j = M$
- The right-hand side of the right equality is almost the slope of the Capital Market Line

CAPM Derivation: the optimality condition for assets i and M (2/3)

$$\frac{\mathrm{E}[R_i] - R_f}{\sigma_{iM}} = \frac{\mathrm{E}[R_M] - R_f}{\sigma_M^2} \qquad \Leftrightarrow \qquad \mathrm{E}[R_i] - R_f = \frac{\sigma_{iM}}{\sigma_M^2}\Big(\mathrm{E}[R_M] - R_f\Big)$$

Remarks:

- Rearranging the equality at left yields an important asset pricing equation at right
- The excess return on asset i is proportional to the excess return on the portfolio
- The constant of proportionality is asset i's relative contribution to portfolio variance
- The constant of proportionality is exactly $\beta_i \coloneqq \sigma_{iM}/\sigma_M^2$, a measure of systematic risk
- Only systemic risk is priced, because idiosyncratic risk has been diversified away

CAPM Derivation: the capital asset pricing equation (3/3)

$$\underbrace{\mathrm{E}[R_i] - R_f}_{\text{Excess Return}} = \underbrace{\beta_i}_{\text{Quantity of Risk}} \times \underbrace{\left(\mathrm{E}[R_M] - R_f\right)}_{\text{Risk Premium}}$$

Remarks:

- The above pricing equation is the key result from the Capital Asset Pricing Model
- CAPM predicts that excess return on asset i is proportional to asset i's market beta
- Assets with high β add to market volatility, and investors demand higher returns
- Investors demand higher returns on assets that contribute to systematic risk
- In the CAPM context, β is always defined with respect to the market portfolio

Question 12

In which industry do you think firms have higher beta values on average: utilities or computers?

A. utilities

B. computers

Remarks:

- betas vary widely across industries, depend on systematic exposure of the industry
- estimated industry betas are available on webpage of NYU's Aswath Damodaran here

Question 12

In which industry do you think firms have higher beta values on average: utilities or computers?

A. utilities

B. **computers**

Remarks:

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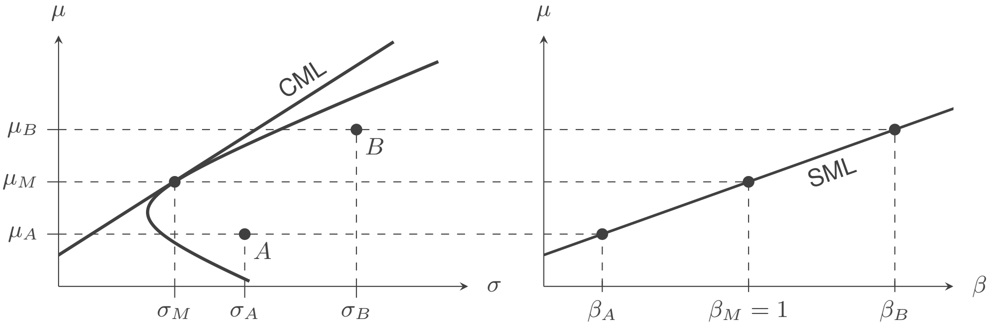
Lecture 09: Capital Asset Pricing Model II
Overview of Topics

Reading: Hillier et al. (2016, Ch 10, & 12), Bodie et al. (2014, Ch 9)

Securities Market Line in Theory and Data



Remarks:

- The securities market line (SML, right) plots expected returns against beta
- A special property of the SML: when an asset has $\beta = 1$, its return equals μ_M
- High beta stocks like asset B have $\beta > 1$ and covary more with the market

Lecture 09: Capital Asset Pricing Model II
Overview of Topics

Reading: Hillier et al. (2016, Ch 10, & 12), Bodie et al. (2014, Ch 9)

Short sellers profit from falling prices

Shorting a stock:

1. Investor borrows a stock at time t from a stock broker
2. Investor sells the stock on the market, receives current price P_t
3. Investor buys the stock back from the market at later price P_{t+1}
4. Investor returns the borrowed stock and earns $P_t - P_{t+1}$

Remarks:

- In CAPM, weights on an asset can be negative, $w_i < 0$, by short selling the asset.
- In practice, short selling is limited as the cost of borrowing stocks can be large.
- Furthermore, regulations prevent most equity mutual funds from shorting stocks

CAPM Extension: Fama-French Three Factor Model

$$\mathrm{E}[R_i] - R_f \quad = \quad \beta_i \times \underbrace{\left(\mathrm{E}[R_M] - R_f\right)}_{\text{Market Factor}} \quad + \quad \gamma_i \times \underbrace{\mathrm{E}[SMB]}_{\text{Size Factor}} \quad + \quad \theta_i \times \underbrace{\mathrm{E}[HML]}_{\text{Value Factor}}$$

Notation and remarks:

- SMB small-minus-big factor; small stocks tend to offer high returns
- HML high-minus-low factor; high book-to-market stocks tend to offer high returns
- γ_i, θ_i Asset i's loadings on the SMB and HML factors, respectively
  - Market factor doesn't fully explain the cross-section of stock returns
  - Fama and French (1992) add two new factors to the CAPM pricing equation
  - Market, size, and value factors explain most cross-sectional variation in returns

Lecture 09: Capital Asset Pricing Model II

Revision Checklist

- ☐ Assumptions of the capital asset pricing model
- ☐ Deriving the capital asset pricing model
- ☐ Securities market line
- ☐ Empirical failures of the capital asset pricing model

Reading: Hillier et al. (2016, Ch 10, & 12), Bodie et al. (2014, Ch 9)

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Lecture 10: Market Efficiency I
Overview of Topics

Reading: Hillier et al. (2016, Ch 13, & 13), Bodie et al. (2014, Ch 11)

Market Efficiency

Speed: Do prices change quickly in response to new relevant information?

Accuracy: Do prices change precisely in response to new relevant information?

Remarks:

- New information reaches markets unpredictably, and surprises market participants
- Perfect efficiency requires immediate and precise price adjustment as news arrives
- In practice, investors react slowly, and sometimes over- or under-react to news
- But professional investors have strong incentives to spot and exploit inefficiencies

Does this group of students use information efficiently? (1/2)

Question 13

I will toss a coin three times. How much would you pay for a risky security that gives you \$100 if I toss at least two heads and \$0 otherwise?

Remarks:

- After you tell us the price you'd pay for the above asset, I will toss the coin once
- I'll tell you the result of the coin toss, and ask you to again price the above asset
- Based on the arrival of new information, we should see a change in willingness to pay

Does this group of students use information efficiently? (2/2)

Question 14

I have tossed the coin once and informed you of the result. Two tosses now remain. With your new information about the first toss, how much would you now pay for the security?

Remarks:

- You've stated the price you'd pay for the asset before any uncertainty was resolved
- Now some uncertainty is resolved: I have tossed once and told you the outcome
- Based on this new information, does your willingness to pay for this asset change?

Price Formation

- Market prices are not established by consensus amongst *all* investors, but rather by those investors who *actively* trade
- Active trades will occur when individual investors receive new information that changes their valuation of an asset
- Upshot of this is that the market may only be efficient with respect to a particular info set e.g. past prices, public information

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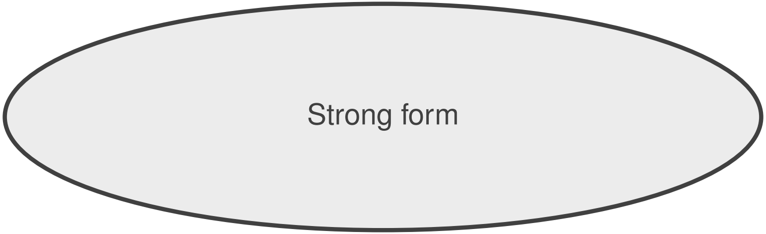
Remarks:

- In practice, investors have incomplete information about traded securities
- How, then, can financial market prices fully reflect all relevant, available info?
- Investors constantly look for new, price-relevant information to base trades on

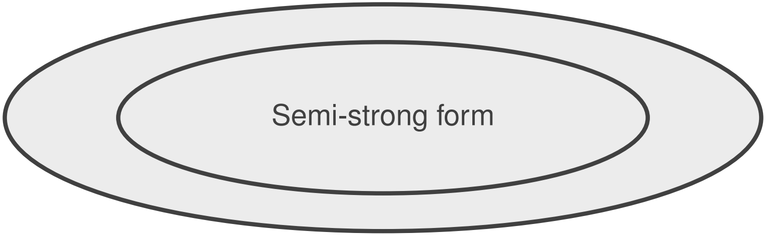
Three forms of the efficient market hypothesis



Remarks:

- The shaded region above represents a specific set of price-relevant information
- New information may lie outside one investor's information set, but inside another's
- The better informed are investors, the more quickly and precisely prices will adjust

Three forms of the efficient market hypothesis



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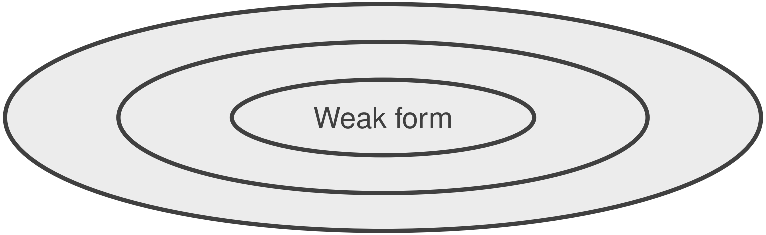

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Strong-form of the efficient market hypothesis

Definition 9

Strong-form EMH:	Stock prices capture all information about a firm, including information only available to company insiders. This is an extreme version of the hypothesis, and probably doesn't hold in the real world.

Remarks:

- In markets with strong efficiency, you could never earn abnormal returns
- Real markets probably aren't strongly efficient, insiders may earn abnormal returns
- However, regulators aim to prevent this insider trading from occurring

Semi-strong-form of the efficient market hypothesis

Definition 10

Semi-strong-form EMH:	Stock prices reflect all public information about a firm, including past prices, fundamental data on the firm's products, patents, balance sheet, earning forecasts, etc.

Remarks:

- Fundamental analysis based on publicly-available information is ineffective
- Price already reflects data in company reports and analyst forecasts, for instance
- In markets with semi-strong efficiency, strong information earns abnormal returns

Weak-form of the efficient market hypothesis

Definition 11

Weak-form EMH:	Stock prices already reflect all information that can be derived by examining market trading data such as the history of past prices, trading volume, or short interest.

Remarks:

- information from past market transactions is already reflected in current prices
- Prices already reflect any technical analysis based on past prices and volumes
- In weakly-efficient markets, semi-strong or strong information earns abnormal returns

Lecture 10: Market Efficiency I
Overview of Topics

Reading: Hillier et al. (2016, Ch 13, & 13), Bodie et al. (2014, Ch 11)

Testing the EMH: simulated trading strategies

- One test of market efficiency is to test whether a specific trading rule would have produced profitable returns in the past
- Problems with testing this strategy
  - “profitable” means relative to some benchmark: which benchmark?
  - really testing a joint hypothesis (see seminar questions)
  - must ensure investment strategy is based on information that was actually available at the time the securities were bought or sold
  - must include costs of acquiring relevant information, costs of buying and selling securities, and any capital gains tax incurred
  - need to determine whether any abnormal return is statistically significant, or simply due to chance

Testing the EMH: informed investor strategies

- If the market prices securities on basis of public information only, then investors with relevant private information should be able to make abnormal returns
- If, on the other hand, the market is efficient, then investors who believe they have identified mis-priced securities are wrong
- These investors are using incomplete information and cannot earn abnormal returns
- To assess efficiency of market:
  - measure the performance of best-informed investors, compare with benchmark
  - investors need superior information before they exhibit abnormal performance
  - these best-informed investors, if they exist, are likely to be professional

Lecture 10: Market Efficiency I
Overview of Topics

Reading: Hillier et al. (2016, Ch 13, & 13), Bodie et al. (2014, Ch 11)

Summary (1/2)

- In a perfectly efficient market, security prices reflect all relevant, available information
  - there are no mis-priced securities
  - security prices are consistent with some (unspecified) pricing rule e.g. CAPM
- Different forms of market efficiency, depending on what type of information is captured by security prices
  - Weak form implies security prices reflect past prices
  - semi-strong form implies security prices reflect all publicly-available information
  - strong form implies security prices reflect all information, both public and private

Summary (2/2)

- If market is fully efficient
  - security prices respond quickly and accurately to receipt of new, relevant information
  - changes in expected return are driven by changes in risk-free rate and/or changes in risk premia
  - changes in security prices driven by other events are necessarily random
  - impossible to discriminate between profitable and unprofitable investments on the basis of currently-available information
  - no statistically-significant difference on average between performance of informed and uninformed investors, or between different groups of informed investors

Lecture 10: Market Efficiency I

Revision Checklist

- ☐ Forms of Market Efficiency
- ☐ Two EMH Testing Strategies
- ☐ Summary

Reading: Hillier et al. (2016, Ch 13, & 13), Bodie et al. (2014, Ch 11)

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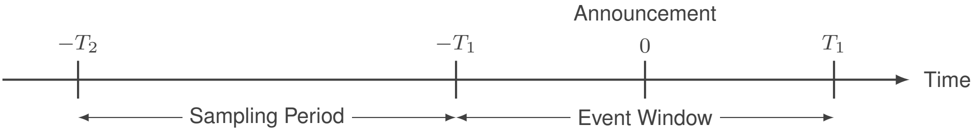
Lecture 11: Market Efficiency II
Overview of Topics

Reading: Hillier et al. (2016, Ch 13), Bodie et al. (2014, Ch 11), Basu (1977), Carhart (1997), Fama, Fisher, et al. (1969), Jensen (1968), Rendleman et al. (1982), Jegadeesh and Titman (1993)

Event studies: measuring market responses to price-relevant events



Remarks:

- Numerous studies have examined the market's reaction to particular events
- Such events could be announcement of earnings, dividends, or stock splits
- Such studies involve estimating abnormal returns around time of announcement
- Abnormal return equals difference between actual return and expected return
- Expected returns assume a particular (possibly incorrect) pricing model, e.g. CAPM

Abnormal Returns (AR)

$$\mathrm{E}[R_i|R_M] - R_f = \alpha_i + \beta_i(R_M - R_f)$$

$$AR_i = R_i - \mathrm{E}[R_i|R_M]$$

Notation and remarks:

$\mathrm{E}[R_i|R_M]$ expected return for firm i conditional on market return

AR_i abnormal return for asset i, unexpected return conditional on market return

- Computing expected returns requires a model—could use any, choose CAPM
- Estimate the model using data from the sampling period $[-T_2, -T_1]$
- Interpret positive abnormal returns as the reward for superior information
- Note that forecastable deviations from CAPM (i.e. α_i) are not abnormal

Cumulative Abnormal Returns (CAR)

$$CAR_i(T) = \sum_{t=-T_1}^{T} AR_{it}$$

Notation and and remarks:

$CAR_i(T)$ Cumulative abnormal return for asset i over interval $[-T_1, T]$, $-T_1 < T < T_1$.

- $CAR_i(T)$ aggregates abnormal returns over time for individual assets
- $CAR_i(T)$ can be plotted for different values of T over the event window
- Problem: idiosyncratic asset-specific response may cause $CAR_i(T) \neq 0$

Cumulative Average Abnormal Returns ($CAAR$)

$$CAAR(T) = \frac{1}{N} \sum_{i=1}^{N} CAR_i(T)$$

Notation and remarks:

$CAAR(T)$ Cumulative average abnormal returns, $CAR_i(T)$ averaged over N assets

- Average $CAR_i(T)$ over assets i to eliminate the idiosyncratic responses
- For averaging, assets must have the single event of interest in common
- Averaging also helps eliminate other firm-specific events within window

Lecture 11: Market Efficiency II
Overview of Topics

Reading: Hillier et al. (2016, Ch 13), Bodie et al. (2014, Ch 11), Basu (1977), Carhart (1997), Fama, Fisher, et al. (1969), Jensen (1968), Rendleman et al. (1982), Jegadeesh and Titman (1993)

Fama, Fisher, et al. (1969): Reaction to Stock Splits (2/2)

- Stock split is interpreted as signal of impending increase in dividend per share
- Increase in dividend per share signals rise in firm’s earning power
- CAAR increased up to 30 months before announcement—why?
  - Stocks may split because of prior share price rise, not vice versa
  - Market may anticipates announcement of stock split

Rendleman et al. (1982): Reaction to Earnings Reports (2/2)

- Rendleman et al. (1982) studies market reaction to quarterly earnings reports
  - firms separated into 10 groups on basis of how recent earnings compare with earnings predicted on basis of past earnings
  - also evidence of sizable reaction in immediate vicinity of event
  - also evidence of delayed reaction up to 90 days when earnings good / bad
  - inconsistent with semi-strong efficiency: market response seems too slow

Jegadeesh and Titman (1993): Secondary Reactions to Earnings

- Jegadeesh, N., & Titman, S. (1993). Returns to buying winners and selling losers: Implications for stock market efficiency. *The Journal of Finance*, *48*(1), 65–91
  - classify stocks as winners or losers based on past returns, measure subsequent relative performance of winners and losers
  - findings: market ignores that good earnings foretell future good earnings—in other words, that firms exhibit momentum
  - once market catches on, it over-reacts by interpreting a succession of good reports as a precursor of many more to follow
  - market is then surprised at unexpectedly bad reports of past winners
- Findings support hypothesis that firms quickly revert to long-run mean in terms of relative growth rates in reported earnings per share

Lecture 11: Market Efficiency II
Overview of Topics

Reading: Hillier et al. (2016, Ch 13), Bodie et al. (2014, Ch 11), Basu (1977), Carhart (1997), Fama, Fisher, et al. (1969), Jensen (1968), Rendleman et al. (1982), Jegadeesh and Titman (1993)

Do Professionals Earn Abnormal Returns?

- If market is strong-form efficient, and CAPM holds, then all securities lie on Securities Market Line (SML)
  - Assuming estimates of beta and expected return are based on state-of-the-art analysis of all relevant available information
  - any analyst who reckons a particular security lies off the SML is basing his estimates on less than complete information
- For any given group of investors, errors in sample estimates of beta and expected return should result in portfolios clustering on either side of SML
- On average, portfolios neither sit above nor below the SML

Jensen (1968): Performance of Mutual Funds (2/2)

- Jensen (1968) examined performance of 115 US mutual funds over 1955-1964
  - S&P 500 used as proxy for market portfolio
  - beta computed by regressing return on fund against return on S&P 500 index
  - abnormal return measured relative to SML
  - SML: passive “buy and hold” investment strategy that requires no special info
- Many more funds are positioned below SML than above it
  - on average, mutual funds under-performed benchmark by ca. 1% per annum
  - ignoring fees & expenses, avg performance was no better than benchmark

What About Individual Fund Managers?

- Are there “star” fund managers who consistently outperform the benchmark?
- Jensen (1968): once managers’ fees have been added back in, the number of funds exhibiting statistically-significant positive or negative out-performance is roughly equal to what would be expected on the basis of chance alone
- Carhart (1997):
  - studied all US funds from 1962 to 1993, including ones that did not survive
  - found that mutual fund managers, on average, do possess significant skills
  - best performers in one year continue to do well for following three years

Lecture 11: Market Efficiency II

Revision Checklist

- ☐ Event Studies and Abnormal Returns
- ☐ Empirical Evidence on Market Reactions
- ☐ Market Efficiency and Professional Investing

Reading: Hillier et al. (2016, Ch 13), Bodie et al. (2014, Ch 11), Basu (1977), Carhart (1997), Fama, Fisher, et al. (1969), Jensen (1968), Rendleman et al. (1982), Jegadeesh and Titman (1993)

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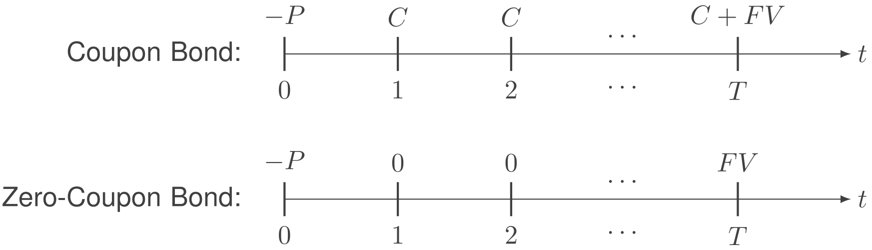
Lecture 12: Bond Pricing I
Overview of Topics

Reading: Hillier et al. (2016, Ch 5, & App 5A), Bodie et al. (2014, Ch 5, 15, & 16)

Bond offers future cash flows in exchange for payment today



Notation and remarks:

P,C,FV Bond price, coupon payment, and face value payment, respectively

- Coupon bonds pay periodic coupons and make a terminal face value payment
- Zero coupon bonds pay no coupons but make a terminal face value payment
- Assume all future coupon and face value payments are certain—no default risk

Think of coupon bonds as portfolios of single payments (1/2)

One Coupon-Paying Bond:

- One coupon bond paying $C = 100$ at $t = 1, 2, \ldots T$ and $FV = 1000$ at $t = T$
- One unique yield y equates the discounted C's and FV with the bond's price

Portfolio of Zero-Coupon Bonds:

- $T + 1$ zero-coupon bonds paying $FV = 100$ at $t = 1, 2, \ldots, T$ and $FV = 1000$ at $t = T$
- There are T different yields that each equate a discounted FV with a bond price

---

Remarks:

- These payment streams are identical and risk-free and should have identical prices
- If these payment streams had different prices, a risk-free profit would be possible

Think of coupon bonds as portfolios of single payments (2/2)

$$P = \sum_{t=1}^{T} \frac{100}{(1+y)^t} + \frac{1000}{(1+y)^T} = \sum_{t=1}^{T} \frac{100}{(1+R_t)^t} + \frac{1000}{(1+R_T)^T}$$

Notation and remarks:

- P current price of a coupon bond
- T total number of cash flows, i.e. the maturity of the bond
- C coupon payment, often expressed as rate, i.e. fraction of face value
- y yield to maturity, i.e. discount rate that equates price and present value
- R_t yield to maturity on a zero-coupon bond maturing in t, called a spot rate

Spot rate: a simple definition

Definition 12

Spot Rate: The yield to maturity on a zero-coupon bond maturing in period t is called the time-t spot rate, denoted R_t.

Remarks:

- Bonds with the same risks but different terms to maturity often differ in yield
- There is a structure to the differences, called the term structure or yield curve
- Spot rates will be useful later for spotting arbitrage opportunities in bond prices

Lecture 12: Bond Pricing I
Overview of Topics

Reading: Hillier et al. (2016, Ch 5, & App 5A), Bodie et al. (2014, Ch 5, 15, & 16)

The pure yield curve: a simple definition

Definition 13

Pure yield curve:	A plot of yields to maturity against time to maturity for zero-coupon treasury securities of different maturities.

Remarks:

- The yield curve represents visually the term structure of interest rates
- We can represent the yield curve using spot rates, discount factors, or forward rates
- All three yield curve representations contain the same information about term structure

Yield curve via spot rates

$$P = PV = \frac{FV}{(1 + R_T)^T}$$

Notation and remarks:

- P price of a zero-coupon bond paying FV in period T for certain
- PV present value of zero-coupon bond face value FV paid in T
- R_T spot rate used to discount cash flows that arrive in period T
- ► Spot rates vary depending on the maturity T of the relevant cash flow
- ► Note that T here represents a *future maturity date*, not the current period
- ► The yield curve, or term structure, is the set of spot rates at different maturities
- ► The yield curve is the set $\{R_1, R_2, \ldots, R_T\}$ for different maturities

Use observed ZCB prices to calculate and plot the yield curve

For ZCBs with $FV = 100$:

Price (in \$)	Maturity (in yrs)	Spot Rate (in %)
70.0	10	3.63
73.0	09	3.56
76.0	08	3.49
80.0	07	3.24
83.0	06	3.15
86.0	05	3.06
90.0	04	2.67
93.0	03	2.45
96.0	02	2.06
99.0	01	1.01

Remarks:

- the yield curve here is upward sloping
- the spot rates here rise with maturity
- see real-world yield curves in the FT

Using spot rates to discount cash flows on different dates

$$PV = \frac{C}{1 + R_1} + \frac{C}{(1 + R_2)^2} + \cdots + \frac{FV + C}{(1 + R_T)^T}$$

Notation and remarks:

- Use the spot rates defined above to discount cash flows from a coupon bond
- If the coupon bond is appropriately priced, PV should equal the bond's price P
- If price P differs from PV using spot rates, you have an arbitrage opportunity
- We discuss yield curve arbitrage in detail in the next lecture

Yield curve via discount factors

$$D_t = \frac{1}{(1 + R_t)^t}$$

Notation and remarks:

D_t The t-period discount factor, i.e. the present value of one unit of period-t cash

- The t here represents a *future maturity date*, not the current time period
- We have now generalized the definition of discount factor given in lecture one
- The generalization lets us use different rates to discount cash from different dates
- Now, the yield curve can be written as a set of discount factors $\{D_1, D_2, \ldots, D_T\}$.

Using discount factors to discount cash flows on different dates

$$PV = C \cdot D_1 + C \cdot D_2 + \cdots + (100 + C) \cdot D_T$$

Remarks:

- Use the discount factors defined above to discount cash flows from a coupon bond
- These discount factors represent the same yield curve as the spot rates
- Next we'll discover a third representation of the yield curve: forward rates

Yield curve via forward rates

$$(1 + R_t)^t (1 + {}_t f_{t+1}) = (1 + R_{t+1})^{t+1}$$

Notation and and remarks:

${}_t f_{t+1}$ forward rate, i.e. yield on a zero-coupon bond bought at t and maturing at $t+1$

- this equation captures an arbitrage argument: ${}_t f_{t+1}$ equalizes LHS and RHS
- LHS: invest one period at spot rate, reinvest one period at forward rate
- RHS: invest two periods at two-period spot rate
- The yield curve can be written as a set of forward rates $\{{}_0 f_1, {}_1 f_2, \ldots, {}_{T-1} f_T\}$

Summary: the yield curve three ways

1. Spot rates: $\{R_1,\ R_2,\ \ldots,\ R_T\}$
2. Discount factors: $\{D_1,\ D_2,\ \ldots,\ D_T\}$
3. Forward rates: $\{{}_0f_1,\ {}_1f_2,\ \ldots,\ {}_{T-1}f_T\}$

Remarks:

- These yield curve representations describe the structure of interest rates over time
- View bonds with same risks but different maturities as trading in different markets
- Spot rates differ across maturities because maturity markets differ in supply / demand

Lecture 12: Bond Pricing I
Overview of Topics

12.1. Coupon and zero-coupon bonds

12.2. The yield curve three ways

12.3. Two-period forward rate

12.4. Multi-period forward rate

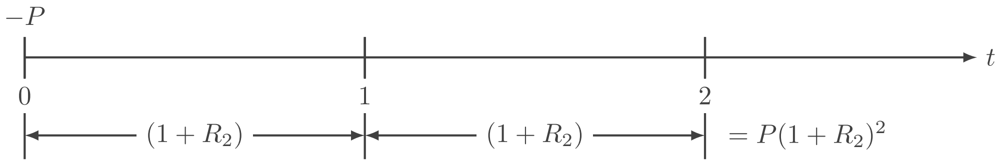
Reading: Hillier et al. (2016, Ch 5, & App 5A), Bodie et al. (2014, Ch 5, 15, & 16)

Spot and forward rates: 2-period geometric example

Remarks:

- Under certainty, two different investment strategies should yield the same payoff

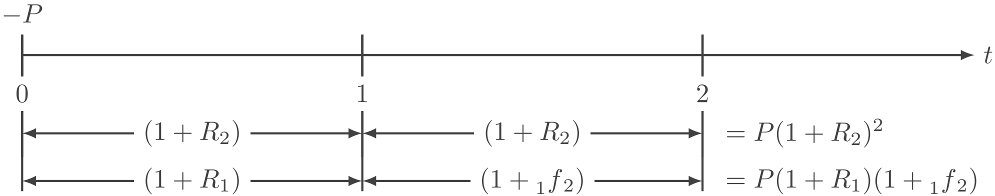
Spot and forward rates: 2-period geometric example



Remarks:

- Under certainty, two different investment strategies should yield the same payoff
- Strategy one: invest at the spot rate of R_2 for two years

Spot and forward rates: 2-period geometric example



Remarks:

- Under certainty, two different investment strategies should yield the same payoff
- Strategy one: invest at the spot rate of R_2 for two years
- Strategy two: invest at spot R_1 for one year, then at the forward rate ${}_1f_2$
- If payoffs differ under certainty, then an arbitrage opportunity exists!

Spot and forward rates: 2-period numeric example

Question 15

The annual yields on one-year and two-year zero-coupon bonds today are 6% and 7%, respectively. In the absence of arbitrage, what must be the annual yield on a one-year zero-coupon bond purchased one year from today?

A. about 7.5%

B. about 8%

Remarks:

- Investments at 1-yr spot then 1-yr forward rate must equal investment at 2-yr spot rate
- If the investment payoffs differed, you could earn riskless zero-cost profit from them
- We will study strategies for earning riskless zero-cost profit in the next lecture

Spot and forward rates: 2-period numeric example

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Spot and forward rates: 2-period numeric example

Solution 15

Spot rates are given in the problem as $R_1 = 0.06$ and $R_2 = 0.07$. To find the forward rate, use the no-arbitrage condition:

$$
\begin{aligned}
(1 + R_1)(1 + {}_1f_2) &= (1 + R_2)^2 \\
\Leftrightarrow \quad {}_1f_2 &= \frac{(1 + R_2)^2}{1 + R_1} - 1 = \frac{1.07^2}{1.06} - 1 = 0.08009434\,.
\end{aligned}
$$

The one-period spot rate one period into the future, which we denote ${}_1f_2$, will equal about 8%. Hence, the one-period spot rate is expected to rise.

Lecture 12: Bond Pricing I
Overview of Topics

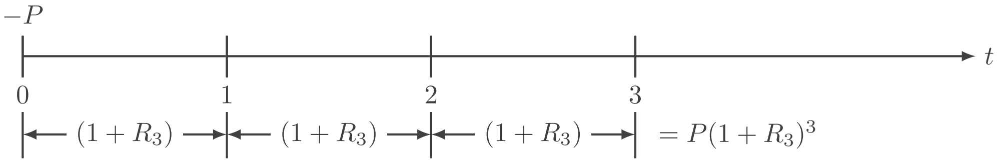
Reading: Hillier et al. (2016, Ch 5, & App 5A), Bodie et al. (2014, Ch 5, 15, & 16)

Multi-period forward rates: a 3-period geometric example

Remarks:

- Under certainty, three different investment strategies should yield the same payoff

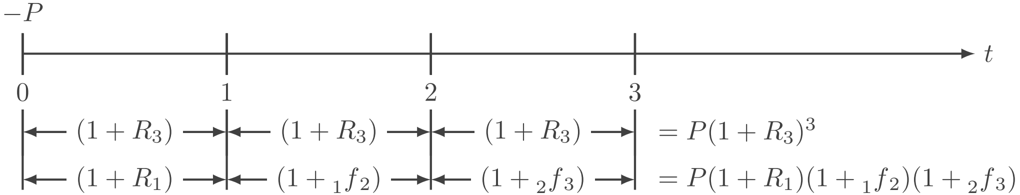
Multi-period forward rates: a 3-period geometric example



Remarks:

- Under certainty, three different investment strategies should yield the same payoff
- Strategy one: invest at the three-year annual spot rate of R_3

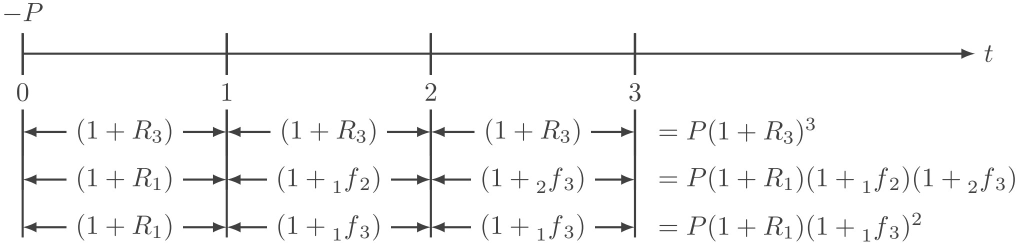
Multi-period forward rates: a 3-period geometric example



Remarks:

- Under certainty, three different investment strategies should yield the same payoff
- Strategy one: invest at the three-year annual spot rate of R_3
- Strategy two: invest at the one-year spot R_1, then forward rate ${}_1f_2$, then ${}_2f_3$

Multi-period forward rates: a 3-period geometric example



Remarks:

- Under certainty, three different investment strategies should yield the same payoff
- Strategy one: invest at the three-year annual spot rate of R_3
- Strategy two: invest at the one-year spot R_1, then forward rate ${}_1f_2$, then ${}_2f_3$
- Strategy three: invest at the one-year spot rate R_1 then two-year forward rate ${}_1f_3$

Spot and forward rates: 3-period numeric example

Question 16

The annual yields on two-year and three-year zero-coupon bonds today are 7% and 8%, respectively. In the absence of arbitrage and uncertainty, what must be the annual yield on a one-year zero-coupon bond purchased two years from today?

A. about 9%

B. about 10%

Remarks:

- Investments at 2-yr spot then 1-yr forward rate must equal investment at 3-yr spot rate
- If the investment payoffs differed, you could earn riskless zero-cost profit from them
- We will study strategies for earning riskless zero-cost profit in the next lecture

Spot and forward rates: 3-period numeric example

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- Investments at 2-yr spot then 1-yr forward rate must equal investment at 3-yr spot rate
- If the investment payoffs differed, you could earn riskless zero-cost profit from them
- We will study strategies for earning riskless zero-cost profit in the next lecture

Spot and forward rates: 2-period numeric example

Solution 16

Spot rates are given in the problem as $R_2 = 0.07$ and $R_3 = 0.08$. To find the forward rate, use the no-arbitrage condition:

$$
\begin{aligned}
(1+R_2)^2(1+{}_2f_3) &= (1+R_3)^3 \\
\Leftrightarrow \quad {}_2f_3 &= \frac{(1+R_3)^3}{(1+R_2)^2} - 1 = \frac{1.08^3}{1.07^2} - 1 = 0.10028125\,.
\end{aligned}
$$

The one-period spot rate one period into the future, which we denote ${}_2f_3$, will equal about 10%. Hence, the one-period spot rate is expected to rise.

Lecture 12: Bond Pricing I

Revision Checklist

- ☐ Coupon and zero-coupon bonds
- ☐ The yield curve three ways
- ☐ Two-period forward rate
- ☐ Multi-period forward rate

Reading: Hillier et al. (2016, Ch 5, & App 5A), Bodie et al. (2014, Ch 5, 15, & 16)

Intermediate Finance
Table of Contents

Lecture 13: Bond Pricing II
Overview of Topics

13.1. Yield curve arbitrage

13.2. Bootstrapping the yield curve

13.3. Three theories of the yield curve

Reading: Hillier et al. (2016, Ch 5, & App 5A), Bodie et al. (2014, Ch 5, 15, & 16)

Arbitrage: a simple definition

Definition 14

Arbitrage: A trading strategy that takes advantage of two or more securities being mispriced relative to each other. Arbitrage involves locking in a riskless profit by simultaneously entering into transactions in two or more securities.

Remarks:

- Any reasonable set of bond prices should exclude arbitrage opportunities
- If two similar bonds make payments on the same future date ...
- ... then both payments should be discounted at the same discount rate.

Source: Hull (2015)

Yield Curve Arbitrage

- The bond market may use different rates to discount payments that are made on different future dates
- But if two bonds of the same credit quality make a payment on the same future date, then both payments should be discounted at the same rate

---

Remarks:

- otherwise, arbitrage profit by going long (i.e. buying) the bond that is cheap
- and simultaneously going short (i.e. selling) the bond that is expensive
- Any “reasonable” set of bond prices should exclude arbitrage opportunities

Yield curve arbitrage: a numerical example (1/3)

Bond	Term	Coupon Rate	Price	Yield
A	3	05.0%	92.44	7.93%
B	3	10.0%	105.49	7.87%
C	3	07.5%	99.02	7.88%

Remarks:

- Bonds A, B and C are coupon bonds that each have face value equal to 100
- Suppose the yield curve is given by the spot rates $\{R_1 = 6\%, R_2 = 7\%, R_3 = 8\%\}$
- Can we check that these bond prices exclude arbitrage opportunities?

Yield curve arbitrage: a numerical example (2/3)

$$P_A = \frac{5}{1+0.06} + \frac{5}{(1+0.07)^2} + \frac{105}{(1+0.08)^3} = 92.44$$

$$P_B = \frac{10}{1+0.06} + \frac{10}{(1+0.07)^2} + \frac{110}{(1+0.08)^3} = 105.49$$

$$P_C = \frac{7.5}{1+0.06} + \frac{7.5}{(1+0.07)^2} + \frac{107.5}{(1+0.08)^3} = 98.96$$

Remarks:

- Market prices should equal cash flows discounted by spot rates as computed above
- Notice that bond C worth less (\$98.96) that the price listed (\$99.02) in the table above
- This gives rise to an arbitrage opportunity: we should short the overpriced bond C

Yield curve arbitrage: a numerical example (3/3)

1. Short $2\times$ Bond C: receive $2 \times \$99.02$, pay $2 \times \$7.5$ coupons
2. Buy bonds A and B: pay $92.44 + 105.49$, receive $\$5 + \$10$ coupons
3. Net profit of arbitrage trade: $2P_C - P_A - P_B = 198.04 - 197.93 = 0.11$

Remarks:

- Above: an arbitrage portfolio with riskless profit based on the mis-pricing of C
- Bonds A and B were correctly priced, but Bond C was over-priced, so short C
- Units $\Rightarrow$ positive profit now and zero future cash flows ($-2 \times 7.5 + 1 \times 5 + 1 \times 10 = 0$)

Lecture 13: Bond Pricing II
Overview of Topics

Reading: Hillier et al. (2016, Ch 5, & App 5A), Bodie et al. (2014, Ch 5, 15, & 16)

Bootstrapping the Yield Curve

Maturity	Coupon Rate	Yield	Price
1	00.0	8.30	92.34
2	08.5	9.20	98.77
3	10.0	9.97	100.07

Remarks:

- Assume now that the yield curve is unknown but the bonds listed above are observed
- We can deduce the yield curve from the three bonds listed above via bootstrapping
- Use the rule that payments on the same date must be discounted at the same rate

Bootstrapping the Yield Curve

- Use the one-period bond price to find one-period spot rate:

$$P_1 = 92.34 = \frac{100}{1 + R_1} \implies R_1 = 0.083$$

- Use the one-period spot rate to find the two-period spot rate:

$$P_2 = 98.77 = \frac{8.5}{1.083} + \frac{108.5}{(1 + R_2)^2} \implies R_2 = 0.0924$$

- Use the same iterative method as above to find the three-period spot rate:

$$P_3 = 100.07 = \frac{10}{1.083} + \frac{10}{(1.0924)^2} + \frac{110}{(1 + R_3)^3} \implies R_3 = 0.108$$

Lecture 13: Bond Pricing II
Overview of Topics

13.1. Yield curve arbitrage

13.2. Bootstrapping the yield curve

13.3. Three theories of the yield curve

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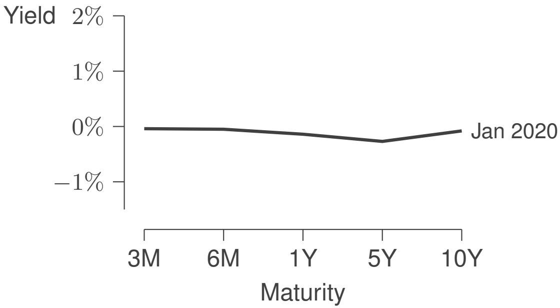
Visualizing term structure stylized facts



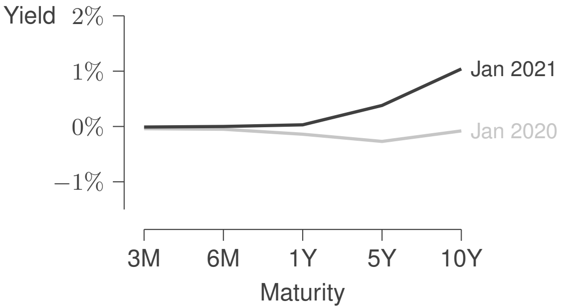
Term Structure Stylized Facts:

1. yields move together across maturities
2. yield curve almost always slopes up
3. high short yield $\Rightarrow$ negative slope likely

Remarks:

- good theories term structure should explain these yield curve stylized facts
- remember that there are plenty of exceptions to these “facts”—hence “stylized”
- this section introduces concepts aimed at explaining the term structure stylized facts

Visualizing term structure stylized facts



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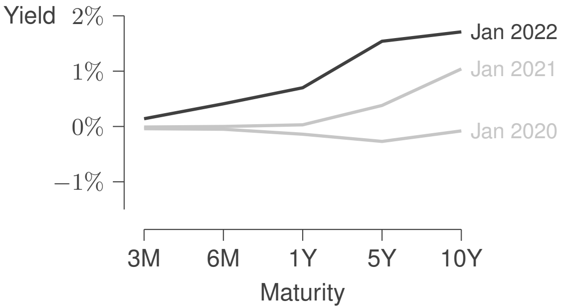

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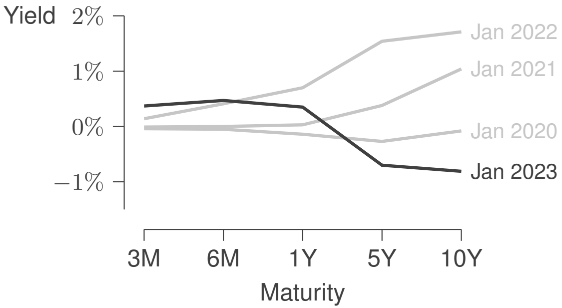

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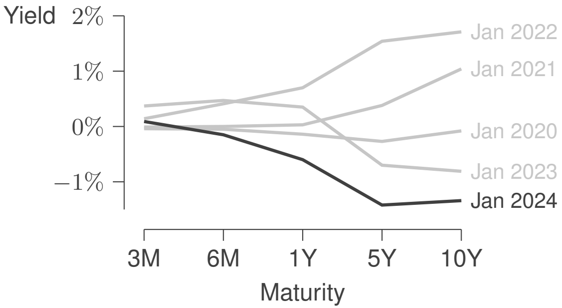

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Three theories to explain the shape of the yield curve.

1. Expectations Hypothesis
2. Liquidity Premium
3. Expected Inflation

Remarks:

- The term structure of interest rates depends on the relative demand for capital of different maturities.
- Lenders supply capital through the form of savings. For example, household deposits, and wealth invested in mutual funds.
- Borrowers demand capital to finance investment opportunities, such as mortgages, and corporate bond issuances.

Theory 1: Expectations Hypothesis

$$(1 + R_t)^t (1 + {}_t f_{t+1}) = (1 + R_{t+1})^{t+1}$$

$$\text{where} \quad {}_t f_{t+s} = \mathrm{E}\Big[{}_t R_{t+s}\Big]$$

Notation and and remarks:

${}_t R_{t+s}$ yield to maturity on zero-coupon bond purchased in t, maturing in $t + s$ periods

- Allow for uncertainty and let forward rates equal expected future spot rates
- In this view, the yield curve will slope upwards if interest rates are expected to rise
- Investors may expect rising rates because of e.g. monetary policy guidance

Theory 2: Liquidity Premium (1/2)

$$(1 + R_t)^t (1 + {}_t f_{t+1}) = (1 + R_{t+1})^{t+1}$$

$$\text{where} \quad {}_t f_{t+s} = \mathrm{E}\Big[{}_t R_{t+s}\Big] + LP$$

Notation and and remarks:

LP Liquidity premium, i.e. premium over expected future spot rate to compensate risk

- Long bonds expose investors to more interest-rate risk due to higher duration
- The yield curve will slope upwards even when rates not expected to rise

Theory 3: Expected inflation

$$R \approx r + \pi_e$$

Notation and remarks:

- R nominal interest rate
- r real interest rate
- π_e expected rate of inflation
- Fisher equation: nominal rate equals real rate plus expected inflation rate
- We've studied the nominal yield curve so far, but investors care about real rates
- Real rates matter because inflation erodes the future value of money
- An upward sloping yield curve can also correspond to a rise in expected inflation

Summary

1. Expectations Hypothesis: forward rates are expected future short rates
2. Liquidity Premium: forward rates include a premium for longer maturities
3. Expected Inflation: nominal rates include a premium for expected inflation

---

Remarks:

- The empirical yield curve is generally upward sloping and prone to parallel shifts
- The empirical yield curve sometimes inverts, especially when short rates are high
- Each of the above theories helps explain these empirical features of the yield curve

Lecture 13: Bond Pricing II

Revision Checklist

- ☐ Yield curve arbitrage
- ☐ Bootstrapping the yield curve
- ☐ Three theories of the yield curve

Reading: Hillier et al. (2016, Ch 5, & App 5A), Bodie et al. (2014, Ch 5, 15, & 16)

Intermediate Finance
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Lecture 14: Forwards and Futures I
Overview of Topics

14.1. Forwards and futures

14.2. Payoff diagrams

14.3. Spot-forward parity

14.4. Forward price arbitrage

Reading: Hillier et al. (2016, Ch 25), Bodie et al. (2014, Ch 22), Hull (2015, Ch 1, 2, 3, 5)

Forward and future contracts let investors hedge risks

Definition 15

Forward and future contracts:	Binding agreements to buy or sell a fixed number of units of an underlying asset at a future expiry date at a price specified upfront when the contract is agreed. No money changes hands at the time the contract is agreed.

Remarks:

- Forwards & futures contracts are derivatives: prices depends on underlying assets
- Notation: payoff depends on forward / future price F_0 and underlying spot price S_T
- Forwards and futures let investors hedge price, interest-rate, exchange-rate risks

Forwards versus Futures

	Future Contract	Forward Contract
Trade:	organized exchange	over-the-counter
Contract:	standardized	tailored
Margin:	standard	negotiated
Risk:	clearinghouse guarantee	couterparty default risk
Settlement:	daily mark-to-market	cash flows occur only at expiry
Regulation:	NFA and CFTC	variable regulation

Remarks:

- A wheat farmer and a miller might negotiate a custom forward contract on wheat price
- Futures markets help to standardize the type of forward arrangement described above

We will find no-arbitrage prices for forwards and futures

Definition 16

No-arbitrage price:	The theoretical price of an asset that is implied by the assumption that the asset cannot be combined with a portfolio of other assets to lock in a riskless profit.

Remarks:

- Arbitrage opportunities arise when assets are mispriced, as we saw in bond pricing
- To profit, go long in the under-priced asset, and go short in the over-priced asset
- As a first approximation, we assume that riskless arbitrage profits cannot be found
- This absence of arbitrage assumption will allow us to price many types of derivative

Long Forward: a numerical example

Question 17

On 1 Sept, the crude oil spot price is $S_0 = \$48.00/$barrel, and you agree to buy 1 barrel of crude oil on 1 Dec at forward price $F_0 = \$50.00$. On 1 Dec, the spot price is $S_T = \$52.00$, you take physical delivery of 1 barrel of crude oil, and pay F_0 in cash to the seller. What is your payoff?

A. $F_0 - S_0 = \$50 - \$48 = \$2$

B. $S_T - F_0 = \$52 - \$50 = \$2$

C. $F_0 - S_T = \$50 - \$52 = -\$2$

- Payoff equals difference between spot rate at expiry S_T and forward rate F_0
- Long forwards act as a hedge against the possibility of rising oil spot prices
- Soon we'll develop a way to finding the forward price F_0 using no-arbitrage

Long Forward: a numerical example

Question 17

On 1 Sept, the crude oil spot price is $S_0 = \$48.00/$barrel, and you agree to buy 1 barrel of crude oil on 1 Dec at forward price $F_0 = \$50.00$. On 1 Dec, the spot price is $S_T = \$52.00$, you take physical delivery of 1 barrel of crude oil, and pay F_0 in cash to the seller. What is your payoff?

A. $F_0 - S_0 = \$50 - \$48 = \$2$

B. $\boldsymbol{S_T - F_0 = \$52 - \$50 = \$2}$

C. $F_0 - S_T = \$50 - \$52 = -\$2$

- Payoff equals difference between spot rate at expiry S_T and forward rate F_0
- Long forwards act as a hedge against the possibility of rising oil spot prices
- Soon we'll develop a way to finding the forward price F_0 using no-arbitrage

Long Forward: a numerical example

Solution 17

By entering a forward agreement to buy an asset (oil) at a forward price $F_0 = \$50$ that is agreed today and paid at time T, you are taking a long position.

The payoff to your long forward at time T will equal the value of the underlying asset S_T minus the forward price F_0 that you have agreed:

$$\text{Long Forward Payoff} = S_T - F_0 = \$52 - \$50 = \$2\,.$$

What determines the forward price F_0? We will show that it should equal a so-called no-arbitrage price...

Lecture 14: Forwards and Futures I
Overview of Topics

Reading: Hillier et al. (2016, Ch 25), Bodie et al. (2014, Ch 22), Hull (2015, Ch 1, 2, 3, 5)

Payoff at expiry date T

Long forward (or futures) position:

- holder pays the forward price F_0 at expiry in cash
- the holder of long position pays the holder of the short position

Short forward (or futures) position:

- holder either pays underlying spot price S_T at expiry in cash (cash-settled)
- or delivers underlying asset (physically settled) to holder of long position

---

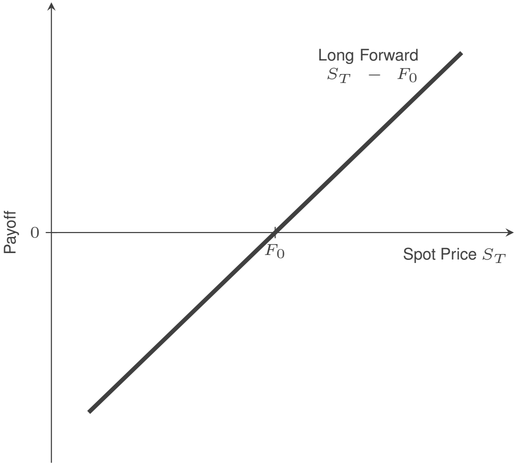
Remarks:

- Payoff to long forward (or futures) position ($=$ buyer of underlying asset) increases with increases in the spot price S_T of the underlying asset
- Payoff to short forward (or futures) position ($=$ seller of underlying asset) decreases with increases in the spot price S_T of the underlying asset

Payoff diagram for long and short forward positions

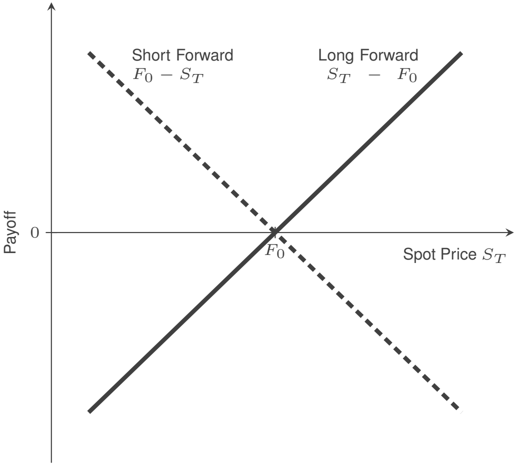
- Payoff diagrams show how payoffs depend on underlying asset prices

Payoff diagram for long and short forward positions



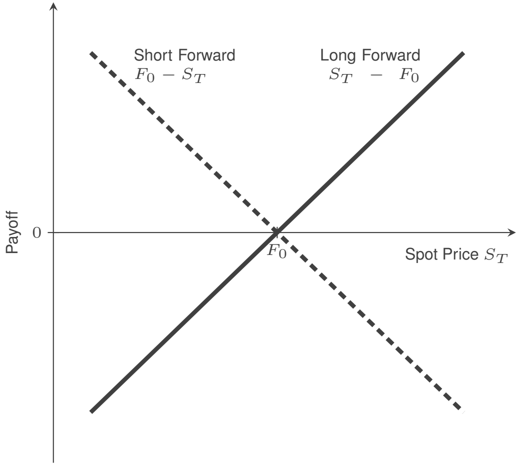
- Payoff diagrams show how payoffs depend on underlying asset prices
- The long forward position yields a payoff of $S_T - F_0$

Payoff diagram for long and short forward positions



- Payoff diagrams show how payoffs depend on underlying asset prices
- The long forward position yields a payoff of $S_T - F_0$
- The short forward position yields a payoff of $F_0 - S_T$

Payoff diagram for long and short forward positions



- Payoff diagrams show how payoffs depend on underlying asset prices
- The long forward position yields a payoff of $S_T - F_0$
- The short forward position yields a payoff of $F_0 - S_T$
- The payoffs to the holders of long and short forward positions are symmetric
- If spot price at expiry equals forward price, the payoff to both parties is zero

Lecture 14: Forwards and Futures I
Overview of Topics

Reading: Hillier et al. (2016, Ch 25), Bodie et al. (2014, Ch 22), Hull (2015, Ch 1, 2, 3, 5)

Spot-forward parity and the no-arbitrage condition (1/3)

How should the fair forward price F_0 be determined?

Suppose we implement the following arbitrage strategy:

1. Borrow S_0 dollars today
2. Buy stock for spot price S_0
3. Enter short forward contract with forward price F_0

Remarks:

- No-arbitrage condition: the cash flow in $t = T$ must equal zero
- Why? The cash outflow at any time $0 \leq t < T$ for this strategy is zero

Spot-forward parity and the no-arbitrage condition (2/3)

Action	Cash Flow at $t = 0$	Cash Flow at $t = T$

Remarks:

- Spot-forward parity: $\$0$ net cash flow at $t = 0$ impies $\$0$ net cash flow at $t = T$
- Be careful: the current spot price is denoted S_0, the spot price at expiry is S_T
- We compound in continuous time—this is common and helpful in derivative pricing

Spot-forward parity and the no-arbitrage condition (2/3)

Action	Cash Flow at $t = 0$	Cash Flow at $t = T$
1. Borrow S_0 dollars:	S_0	$-S_0 e^{RT}$

Remarks:

- Spot-forward parity: $\$0$ net cash flow at $t = 0$ impies $\$0$ net cash flow at $t = T$
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1. Borrow S_0 dollars:	S_0	$-S_0 e^{RT}$
2. Buy stock for S_0:	$-S_0$	S_T
3. Short forward at F_0:	0	$F_0 - S_T$

Remarks:

- Spot-forward parity: $\$0$ net cash flow at $t = 0$ impies $\$0$ net cash flow at $t = T$
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Action	Cash Flow at $t = 0$	Cash Flow at $t = T$
1. Borrow S_0 dollars:	S_0	$-S_0 e^{RT}$
2. Buy stock for S_0:	$-S_0$	S_T
3. Short forward at F_0:	0	$F_0 - S_T$
Total cash flows:	0	$F_0 - S_0 e^{RT}$

Remarks:

- Spot-forward parity: $\$0$ net cash flow at $t = 0$ impies $\$0$ net cash flow at $t = T$
- Be careful: the current spot price is denoted S_0, the spot price at expiry is S_T
- We compound in continuous time—this is common and helpful in derivative pricing

Spot-forward parity and the no-arbitrage condition (3/3)

$$F_0 - S_0 e^{RT} = 0 \quad \Rightarrow \quad F_0 = S_0 e^{RT}$$

Remarks:

- Spot-forward parity: $\$0$ net cash flow at $t = 0$ impies $\$0$ net cash flow at $t = T$
- A $\$0$ investment today yielding positive cash flow in T would constitute arbitrage
- At no initial cost, you can buy stock with borrowed money and enter a forward
- If this trade is costless today, it cannot yield positive cash flows tomorrow

Numerical example: spot-forward parity

Question 18

You want to enter a forward agreement to purchase in 6 months at forward price F_0 a single share in a firm with current spot price $S_0 = \$15.50$. The risk free rate is 2%. What is the fair price of the forward contract, and would your position be long or short?

A. 15.35, short

B. 15.65, long

Numerical example: spot-forward parity

Question 18

You want to enter a forward agreement to purchase in 6 months at forward price F_0 a single share in a firm with current spot price $S_0 = \$15.50$. The risk free rate is 2%. What is the fair price of the forward contract, and would your position be long or short?

A. 15.35, short

B. **15.65, long**

Numerical example: spot-forward parity

Solution 18

By entering a forward agreement to purchase an asset at a forward price F_0 that is agreed today and paid at time T, you are taking a long position.

The no-arbitrage forward price satisfies the spot-forward parity, which states that

$$
\begin{aligned}
F_0 &= S_0 e^{RT} \\
&= 15.5 e^{0.02 \times 0.5} \approx 15.65 \,.
\end{aligned}
$$

If the forward price violates no-arbitrage condition, you could borrow funds and buy the underlying to replicate the forward payoff in T and earn arbitrage profit.

Aside on continuous compounding

Write the T-year compound factor C_{nT} with compound frequency n and yield R as

$$C_{nT} = (1 + R/n)^{nT}.$$

For continuous compounding, let $1/m := R/n$ and take the limit of $n \to \infty$:

$$\begin{aligned}\lim_{n\to\infty} C_{nT} &= \lim_{n\to\infty}(1 + R/n)^{nt} \\ &= \lim_{m\to\infty}(1 + 1/m)^{mRT} \\ &= \left[\lim_{m\to\infty}(1 + 1/m)^{m}\right]^{RT} = e^{RT}.\end{aligned}$$

Remarks:

- Last line of derivation uses the definition of Euler's number $e := \lim_{m\to\infty}(1 + 1/m)^m$
- Continuous compounding is often mathematically more convenient than discrete

Lecture 14: Forwards and Futures I
Overview of Topics

Reading: Hillier et al. (2016, Ch 25), Bodie et al. (2014, Ch 22), Hull (2015, Ch 1, 2, 3, 5)

Forward *over*-pricing and arbitrage opportunities (1/3)

Suppose now $F_0 = \$16$, but all else as before: $S_0 = \$15.50$, $T = 0.5$, $R = 0.02$. The forward is **overvalued** relative to the fair price of $\$15.65$. Arbitrage strategy:

1. Borrow $S_0 = \$15.50$ dollars today
2. Buy stock for spot price $S_0 = \$15.50$
3. Enter short forward contract with forward price $F_0 = \$16$

Cash flow at $t = 0$: $\$0.00$

Cash flow at $t = T$: $\$0.35 = \$16 - \$15.50e^{0.02 \cdot 0.5}$

Remarks:

- Note that the cash outflow at time $t = 0$ for this strategy is zero
- No-arbitrage condition is violated: the cash flow in $t = T$ exceeds zero

Forward *under*-pricing and arbitrage opportunities (2/3)

Suppose now $F_0 = \$14$, but all else as before: $S_0 = \$15.50$, $T = 0.5$, $R = 0.02$. The forward is **undervalued** relative to the fair price of $\$15.65$. Arbitrage strategy:

1. Lend $S_0 = \$15.50$ dollars today
2. Short stock for spot price $S_0 = \$15.50$
3. Enter long forward contract with forward price $F_0 = \$14$

Cash flow at $t = 0$: $\$0.00$

Cash flow at $t = T$: $\$1.66 = \$15.50e^{0.02 \cdot 0.5} - \$14.00$

Remarks:

- Because the forward is underpriced, the arbitrage strategy is exactly reversed
- The principle at work: buy low, sell high; here the forward price is low so you buy

Forward *mis*-pricing and arbitrage opportunities (3/3)

Buy low

Sell high

Remarks:

- In the first example, the forward price F_0 was too high, so we went short (sold)
- In the second example, the forward price F_0 was too low, so we went long (bought)
- If you can perfectly identify mis-priced assets, this strategy yields riskless profit
- Investors exploit it until prices (here, F_0) adjust to eliminate the arbitrage opportunity

Lecture 14: Forwards and Futures I

Revision Checklist

☐ Forwards and futures

☐ Payoff diagrams

☐ Spot-forward parity

☐ Forward price arbitrage

Reading: Hillier et al. (2016, Ch 25), Bodie et al. (2014, Ch 22), Hull (2015, Ch 1, 2, 3, 5)

Intermediate Finance
Table of Contents

Lecture 15: Forwards and Futures II
Overview of Topics

15.1. Short and long calls and puts
15.2. Constructing forwards from options
15.3. Put-call parity
15.4. Put-call arbitrage

Reading: Hillier et al. (2016, Ch 25), Bodie et al. (2014, Ch 20), Hull (2015, Ch 1, 2, 3, & 5)

From forwards to options: an introduction

Forward (or future): contract is a binding obligation to buy or sell:

- a fixed number of units of an underlying asset
- at an agreed expiry date in the future
- at a forward (or futures) price that is specified upfront when the contract is agreed
- no money changes hands at the time the forward (or futures) contract is agreed

Option (call or put): buyer's right but not obligation to buy or sell:

- a fixed number of units of an underlying asset
- at an agreed expiry date in the future
- at a fixed exercise price that is specified upfront when the contract is agreed
- the buyer pays an option premium when the option is written

Options: a simple definition

Definition 17

Option: The right, but not the obligation, to buy or sell an asset at an exercise price on or before an expiry date. The option buyer pays an option premium to the option seller.

Notation and remarks:

- Two types of options: European (exercise at Expiry), American (exercise Anytime)
- The option seller takes a short position, the option buyer takes a long position
- Options to buy the underlying asset are calls, options to sell are puts

P_t Premium (i.e. price) for a European put option at time t

C_t Premium (i.e. price) for a European call option at time t

Payoff diagrams for European long and short calls and puts

Long Call Payoff	$C_T =$	$\max(S_T - X, 0)$
Long Put Payoff	$P_T =$	$\max(X - S_T, 0)$
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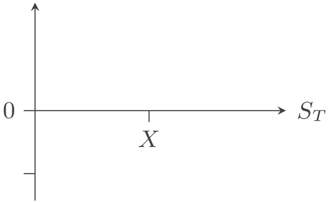
Long Call Payoff



Notation and remarks:

- $\max$ maximization operator, where $\max(A, B)$ equals the greater of A and B
- S_T underlying asset price at expiry $t = T$
- X exercise price specified in option contract
- ► Initial option prices are C_0 and P_0, final payoffs are C_T and P_T (for long position)
- ► Long profit = payoff minus initial price; short profit = initial price minus profit

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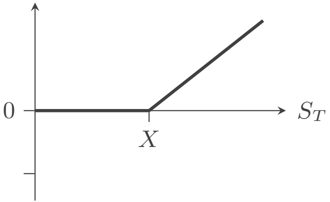

Notation and remarks:

- max: maximization operator, where $\max(A, B)$ equals the greater of A and B
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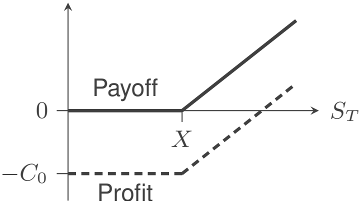
Long Call Payoff and Profit



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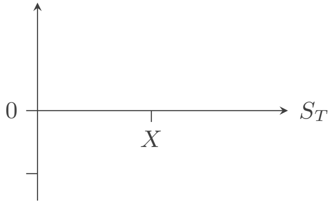

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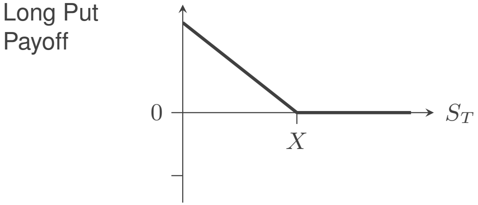

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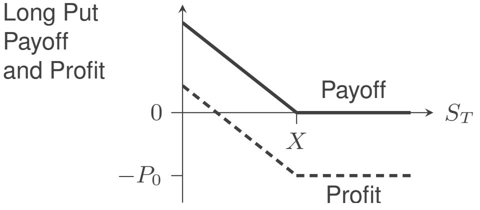

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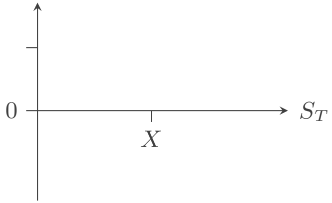
Short Call Payoff



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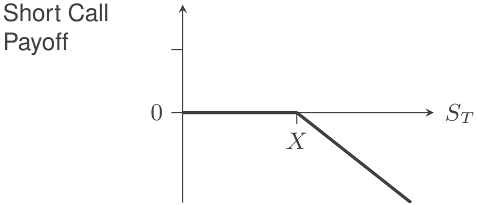

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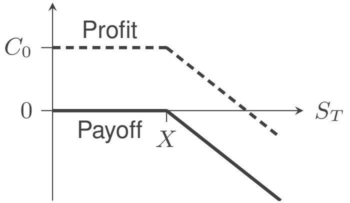
Short Call Payoff and Profit



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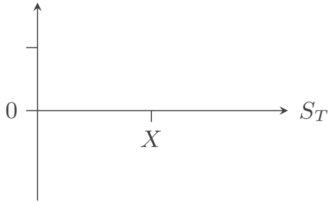
Short Put Payoff



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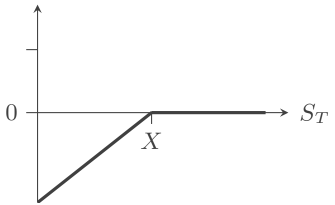

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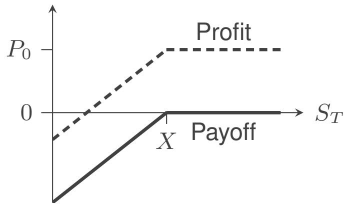
Short Put Payoff and Profit



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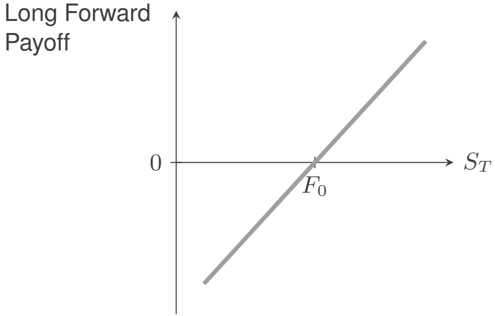
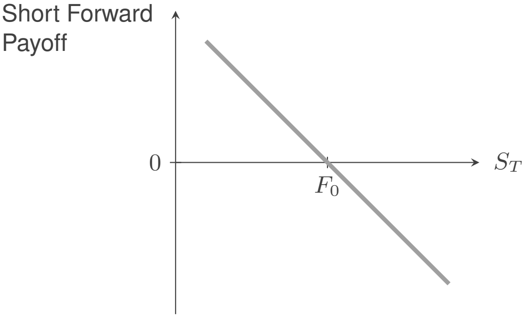
Lecture 15: Forwards and Futures II
Overview of Topics

Reading: Hillier et al. (2016, Ch 25), Bodie et al. (2014, Ch 20), Hull (2015, Ch 1, 2, 3, & 5)

Comparing payoffs on forwards and options

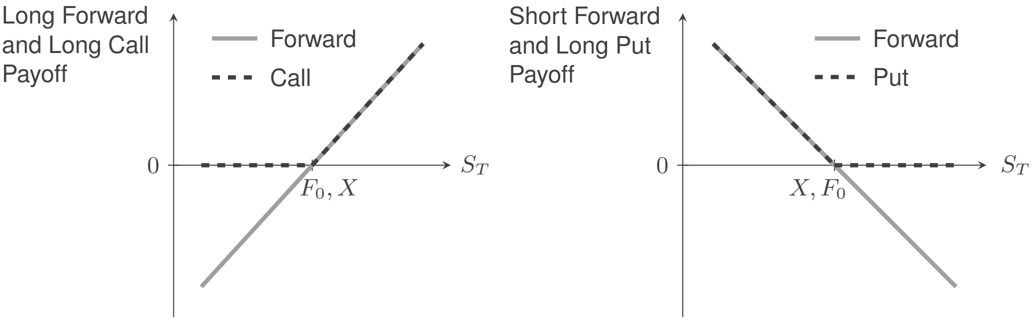




Remarks:

- Forwards represent an obligation to transact; options represent the right to transact
- Their payoff structures therefore differ: options offer protection against downside risk
- The protection comes at a price however; option buys must pay a premium for the right

Comparing payoffs on forwards and options



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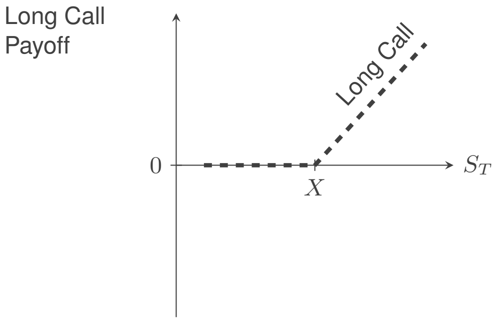
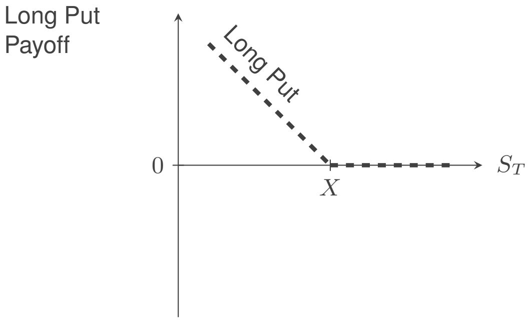
Constructing forward payoffs from option payoffs (1/2)





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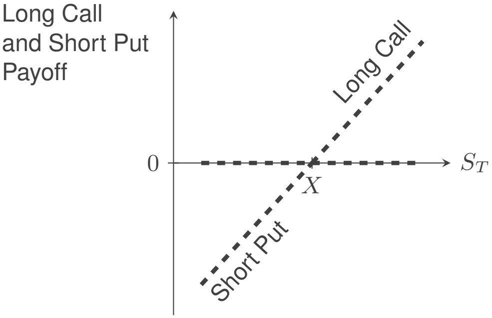

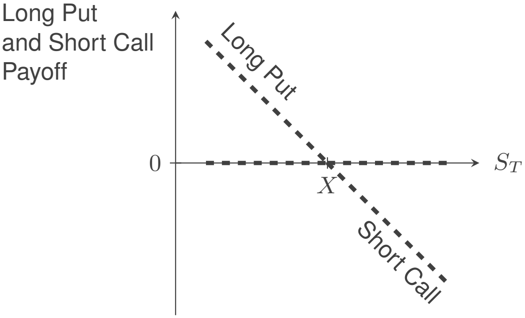

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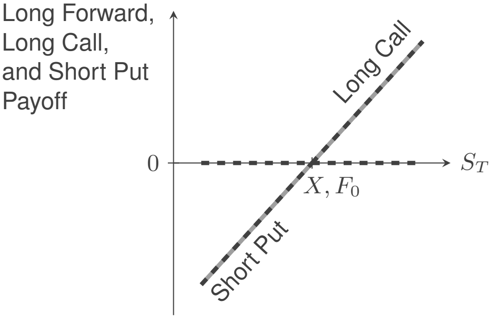

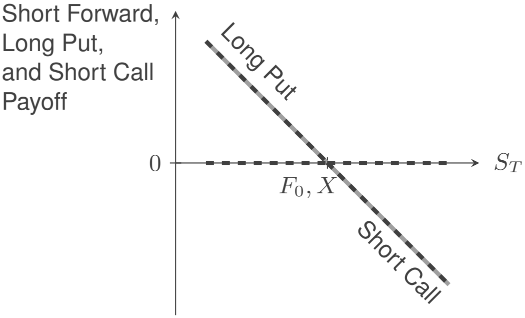

Remarks:

- Forwards represent an obligation to transact; options represent the right to transact
- Their payoff structures therefore differ: options offer protection against downside risk
- The protection comes at a price however; option buys must pay a premium for the right

Constructing forward payoffs from option payoffs (2/2)

$$\begin{aligned}\text{Long forward payoff} &= \text{Long call payoff} + \text{Short put payoff}\\ &= \max(S_T - X, 0) - \max(X - S_T, 0) = S_T - X\end{aligned}$$

$$\begin{aligned}\text{Short forward payoff} &= \text{Long put payoff} + \text{Short call payoff}\\ &= \max(X - S_T, 0) - \max(S_T - X, 0) = X - S_T\end{aligned}$$

Remarks:

- The expressions show that forwards can be constructed from options
- If strike price $X = F_0$, then we recover forward payoffs from option payoffs
- This shows mathematically the relationship the previous slide shows geometrically

Lecture 15: Forwards and Futures II
Overview of Topics

15.1. Short and long calls and puts

15.2. Constructing forwards from options

15.3. Put-call parity

15.4. Put-call arbitrage

Reading: Hillier et al. (2016, Ch 25), Bodie et al. (2014, Ch 20), Hull (2015, Ch 1, 2, 3, & 5)

Put-call parity and the no-arbitrage condition

	Portfolio 1	Portfolio 2
Cost $=$	$C_0 + Xe^{-RT}$	$S_0 + P_0$
Payoff $=$	Call Payoff $+ X$	$S_T +$ Put Payoff
$=$	$\max(S_T - X, 0) + X$	$S_T + \max(X - S_T, 0)$
$=$	$\max(S_T, X)$	$\max(S_T, X)$

Remarks:

- Portfolio 1: buy a European call for C_0 and invest Xe^{-RT} at the risk-free rate
- Portfolio 2: buy a European put for P_0 on a stock and buy the underlying stock
- The two portfolios offer identical payoffs at the expiry date of the options $t = T$
- Because payoffs are equal at expiry, the portfolios must cost the same at $t = 0$

Put-call parity: an option pricing condition

$$C_0 + Xe^{-RT} = S_0 + P_0$$

Remarks:

- Left-hand side is payoff from Portfolio 1, right-hand side is payoff from Portfolio 2
- Put-call parity: the prices of these portfolios must be equal under no-arbitrage
- Put-call parity tells us the price of one option, if we know the price of another
- Useful, but we will also want to price options independently of other options (later)

Put-call parity: a numerical example

Question 19

Suppose the stock price is \$31, the exercise price is \$30, the risk-free rate is 2% per year, the price of a 3-month European call is \$4. If the put-call parity holds, what must be the price of a 3-month European put?

A. 2.85

B. 3.85

Remarks:

- Rearrange the put-call parity condition to get $P_0 = C_0 + Xe^{-RT} - S_0$
- We can use the put-call parity condition to solve the above problem and many others

Put-call parity: a numerical example

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Suppose the stock price is \$31, the exercise price is \$30, the risk-free rate is 2% per year, the price of a 3-month European call is \$4. If the put-call parity holds, what must be the price of a 3-month European put?

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Remarks:

- Rearrange the put-call parity condition to get $P_0 = C_0 + Xe^{-RT} - S_0$
- We can use the put-call parity condition to solve the above problem and many others

Put-call parity: a numerical example

Solution 19

The stock price is given as $S_0 = 31$, the exercise price is given as $X = 30$, the call premium is given as $C_0 = 4$, the risk-free rate is given as $R = 0.02$, and the time to expiry is $T = 0.25$. From the put-call parity,

$$
\begin{aligned}
C_0 + Xe^{-RT} &= S_0 + P_0 \\
\Leftrightarrow \quad P_0 &= C_0 + Xe^{-RT} - S_0 \\
&= 4 + 30e^{-0.02\times 0.25} - 31 = 2.85\,.
\end{aligned}
$$

Lecture 15: Forwards and Futures II
Overview of Topics

15.1. Short and long calls and puts

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Example 1: Option mis-pricing and put-call arbitrage (1/2)

Question 20

Suppose the stock price is \$31, the exercise price is \$30, the risk-free rate is 2% per year, the price of a 3-month European call is \$3.75 and the price of a 3-month European put is \$3.00. What does your arbitrage strategy involve:

A. Long puts and short calls

B. Short puts and long calls

Remarks:

- Recall that if the put-call parity held, we would have $C_0 = 4$ and $P_0 = 2.85$
- For arbitrage, we need to buy low and sell high, and these assets are mis-priced
- In this example, the call looks cheap and the put looks expensive

Example 1: Option mis-pricing and put-call arbitrage (1/2)

Question 20

Suppose the stock price is \$31, the exercise price is \$30, the risk-free rate is 2% per year, the price of a 3-month European call is \$3.75 and the price of a 3-month European put is \$3.00. What does your arbitrage strategy involve:

A. Long puts and short calls

B. **Short puts and long calls**

Remarks:

- Recall that if the put-call parity held, we would have $C_0 = 4$ and $P_0 = 2.85$
- For arbitrage, we need to buy low and sell high, and these assets are mis-priced
- In this example, the call looks cheap and the put looks expensive

Example 1: Option mis-pricing and put-call arbitrage (2/2)

Action	Cash Flow at $t = 0$	Cash Flow at $t = T$
Lend $PV(X)$:	$-Xe^{-RT}$	X
Long Call:	$-C_0$	$\max(S_T - X, 0)$
Short Stock:	S_0	$-S_T$
Short Put:	P_0	$-\max(X - S_T, 0)$
Total:	$P_0 + S_0 - C_0 - Xe^{-RT}$	0

Remarks:

- Strategy yields positive net cash flow immediately and zero future net cash flows
- The positive net cash flow arises because we have bought low and sold high
- The arbitrage profit in period 0 is computed as: $3.00 + 31 - 3.75 - 30e^{-0.02*0.25} = 0.40$

Example 2: Option mis-pricing and put-call arbitrage (1/2)

Question 21

Suppose the stock price is \$31, the exercise price is \$30, the risk-free rate is 2% per year, the price of a 3-month European call is \$4.25 and the price of a 3-month European put is \$2.50.

A. Long puts and short calls

B. Short puts and long calls

Remarks:

- Recall that if the put-call parity held, we would have $C_0 = 4$ and $P_0 = 2.85$
- For arbitrage, we need to buy low and sell high, and these assets are mispriced
- In this example, the call looks expensive and the put looks cheap

Example 2: Option mis-pricing and put-call arbitrage (1/2)

Question 21

Suppose the stock price is \$31, the exercise price is \$30, the risk-free rate is 2% per year, the price of a 3-month European call is \$4.25 and the price of a 3-month European put is \$2.50.

A. **Long puts and short calls**

B. Short puts and long calls

Remarks:

- Recall that if the put-call parity held, we would have $C_0 = 4$ and $P_0 = 2.85$
- For arbitrage, we need to buy low and sell high, and these assets are mispriced
- In this example, the call looks expensive and the put looks cheap

Example 2: Option mis-pricing and put-call arbitrage (2/2)

Action	Cash Flow at $t = 0$	Cash Flow at $t = T$
Borrow $PV(X)$:	Xe^{-RT}	$-X$
Short Call:	C_0	$-\max(S_T - X, 0)$
Long Stock:	$-S_0$	S_T
Long Put:	$-P_0$	$\max(0, X - S_T)$
Total:	$Xe^{-RT} + C_0 - S_0 - P_0$	0

Remarks:

- Strategy yields positive net cash flow immediately and zero future net cash flows
- The positive net cash flow arises because we have bought low and sold high
- The arbitrage profit in period 0 is computed as: $30e^{-0.02*0.25} + 4.25 - 31 - 2.50 = 0.60$

Lecture 15: Forwards and Futures II

Revision Checklist

- ☐ Short and long calls and puts
- ☐ Constructing forwards from options
- ☐ Put-call parity
- ☐ Put-call arbitrage

Reading: Hillier et al. (2016, Ch 25), Bodie et al. (2014, Ch 20), Hull (2015, Ch 1, 2, 3, & 5)

Intermediate Finance
Table of Contents

Lecture 16: Options I
Overview of Topics

Reading: Hillier et al. (2016, Ch 22), Bodie et al. (2014, Ch 21), Hull (2015, Ch 10, 11, 13)

Option pricing: our road map

- At expiry, an option's price must equal the option's payoff—this is trivial
- Before expiry, we can use arbitrage arguments to establish bounds for option prices
- We can also use a simplified binomial model to give us exact European option prices
- Next lecture we find exact prices using the more general Black-Scholes model

Lecture 16: Options I
Overview of Topics

Reading: Hillier et al. (2016, Ch 22), Bodie et al. (2014, Ch 21), Hull (2015, Ch 10, 11, 13)

European Call Option Bounds on Assets with no Dividends (1/2)

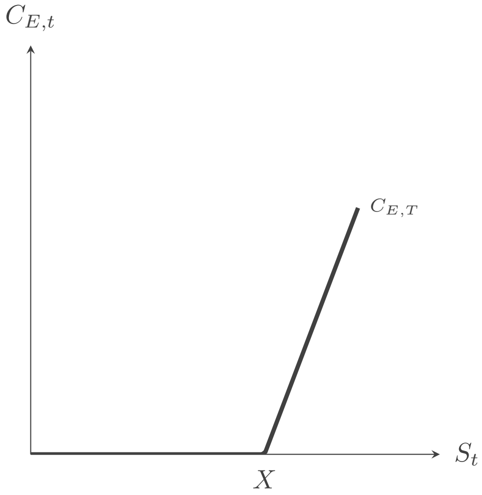
$$C_{E,t} \geq 0$$
$$C_{E,t} \geq S_t - PV(X)$$

Notation and remarks:

$C_{E,t}$ Value of European call on non-dividend paying underlying asset at $t < T$

- First bound: no obligation to exercise
- Second bound: compare two portfolio payoffs
- Portfolio 1: long European call; payoff at expiry: $\max(S_T - X, 0)$
- Portfolio 2: long underlying + borrow $PV(X)$; payoff at expiry: $S_T - X$
- Portfolio 1 pays at least what portfolio 2 pays at expiry: $\max(S_T - X, 0) \geq S_T - X$
- Therefore the call option costs at least what portfolio 2 costs at time $t < T$

European Call Option Bounds on Assets with no Dividends (2/2)



- The thick black line shows the value $C_{E,T}$ of a European call at expiry $t = T$

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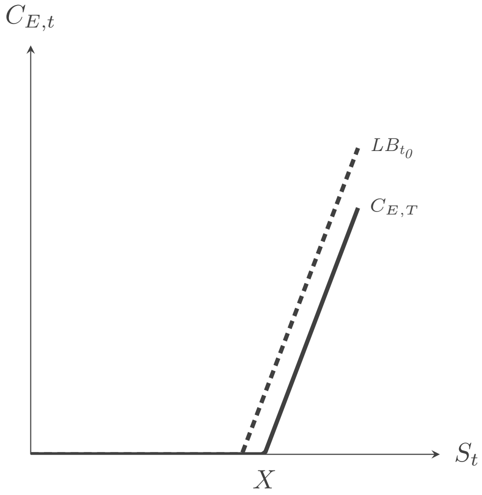

- The thick black line shows the value $C_{E,T}$ of a European call at expiry $t = T$
- Before expiry $t < T$, we have established two lower bounds for $C_{E,t}$
- Combine the two lower bounds above to get

$$C_{E,t} \geq \max\Big(S_t - PV(X), 0\Big) = LB_t$$

- The dashed black line shows the lower bound LB_t for the call value $C_{E,t}$ before expiry $t < T$

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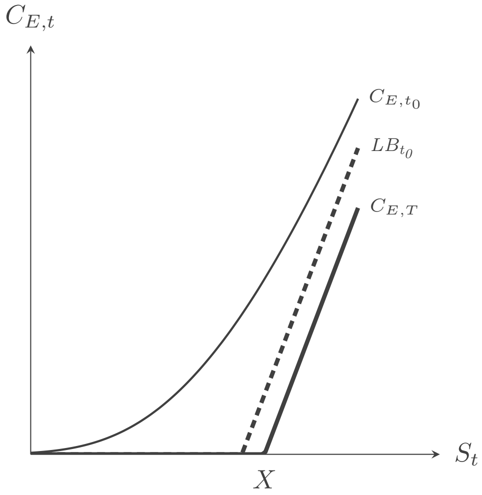

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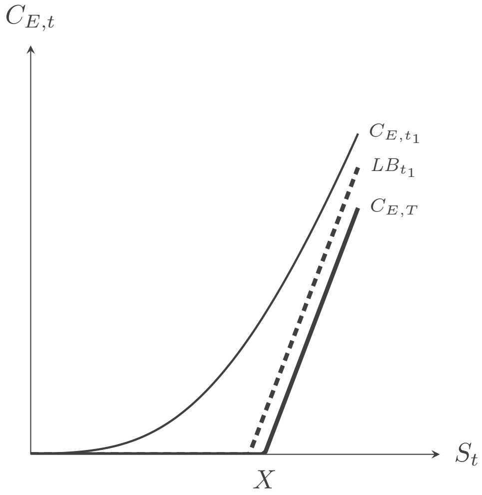

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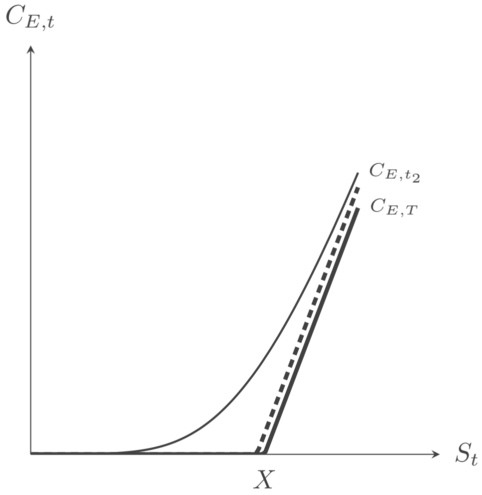

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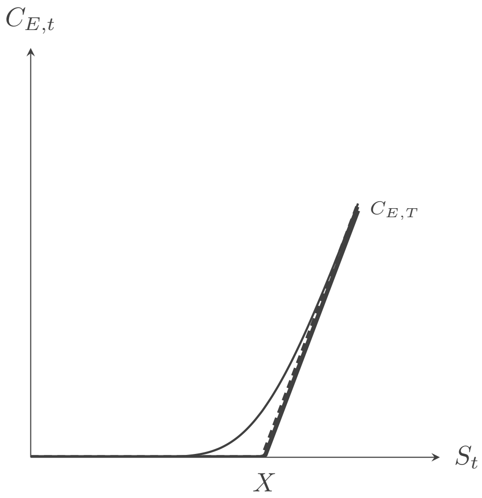

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- As time passes, lower bound LB_t and call value $C_{E,t}$ approach the call value at expity $t = T$

European Put Option Bounds on Assets with no Dividends (1/2)

$$P_{E,t} \geq 0$$
$$P_{E,t} \geq PV(X) - S_t$$

Notation and remarks:

$P_{E,t}$ Value of European put on non-dividend paying underlying asset at $t < T$

- First bound: no obligation to exercise
- Second bound: compare two portfolio payoffs
- Portfolio 1: long European put; payoff at expiry: $\max(X - S_T, 0)$
- Portfolio 2: short underlying asset + lend $PV(X)$; payoff at exipry: $X - S_T$
- Portfolio 1 pays at least what portfolio 2 pays at expiry: $\max(X - S_T, 0) \geq X - S_T$
- Therefore the put option costs at least what portfolio 2 costs at time $t < T$

American Call Option Bounds on Assets with no Dividends

$$C_{A,t} \geq 0$$
$$C_{A,t} \geq S_t - PV(X)$$

Notation and remarks:

$C_{A,t}$ Value of American call on non-dividend paying underlying asset at $t < T$

- An American call option can be exercised before expiry, so its value is greater than or equal to that of a European call, $C_{A,t} \geq C_{E,t}$
- The value of American-style call option whilst it is still 'alive' is strictly greater than the payoff to the option when it is 'dead' (i.e. when it is exercised)
- Early exercise of an American-style call option results in a payoff of $S_t - X$. This is never optimal if the asset pays no dividends
- The value of an American-style call option on non-dividend-paying underlying asset therefore equals that of corresponding European-style call option

American Put Option Bounds on Assets with no Dividends

$$P_{A,t} \geq 0$$
$$P_{A,t} \geq X - S_t$$

Notation and remarks:

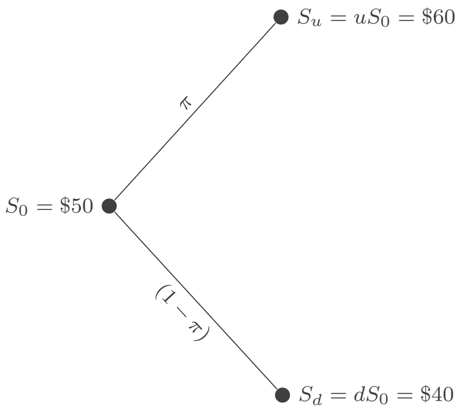
$P_{A,t}$ Value of American put on non-dividend paying underlying asset at $t < T$

- An American put option can be exercised before expiry, so its value is greater than or equal to that of a European put, $P_{A,t} \geq P_{E,t}$
- Unclear whether the value of American-style put option while 'alive' is strictly greater than its value when 'dead' (i.e. when it is exercised)
- Early exercise of American-style put option results in a payoff of $X - S_t$; can be optimal if the option is sufficiently in the money
- The value of an American-style put option on non-dividend-paying underlying asset exceeds that of corresponding European-style put option

Lecture 16: Options I
Overview of Topics

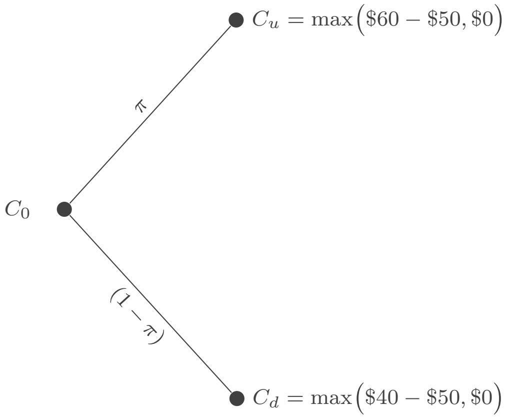
Reading: Hillier et al. (2016, Ch 22), Bodie et al. (2014, Ch 21), Hull (2015, Ch 10, 11, 13)

Binomial tree of the underlying share price



- Assume the share price of a stock follows a discrete binomial distribution
- The figure at left shows how the share price can either jump up or jump down
- Current share price $S_0 = \$50$; future share price either $S_u = \$60$ or $S_d = \$40$
- The two possible future states are the up state and the down state
- These two states occur with (unknown) probabilities π and $1 - \pi$

Binomial tree of the price of a European call on the stock



- Assume the share price of a stock follows a discrete binomial distribution
- Figure at left shows how the call payoff depends on underlying share price
- Current share price $S_0 = \$50$; future share price either $S_u = \$60$ or $S_d = \$40$
- Call payoff in up state with $X = 50$: $\max\big(\$60 - \$50, \$0\big) = \$10$
- Call payoff in down state with $X = 50$: $\max\big(\$40 - \$50, \$0\big) = \$0$

Binomial option pricing using a replicating portfolio (1/2)

$$C_u = \Delta S_u - Be^{RT}$$
$$C_d = \Delta S_d - Be^{RT}$$

Remarks:

- Replicating portfolio: long Δ units of stock $+$ short B units of bond
- Choose Δ and B to replicate the call option payoffs in up and down states
- Call payoffs in the up and down states are denoted C_u and C_d, respectively
- Recall that u and d denote percent changes in share price in up and down states
- Share price in the up and down states: $S_u = uS_0$ and $S_d = dS_0$

Binomial option pricing using a replicating portfolio (2/2)

$$C_0 = \Delta S_0 - B, \qquad \text{where } \Delta = \frac{C_u - C_d}{S_0(u-d)}, \qquad B = \frac{dC_u - uC_d}{(u-d)e^{RT}}$$

Remarks:

- The expressions for Δ and B here solve the arbitrage pricing conditions above
- Buying Δ units of stock and selling B units of bond thus replicates the call payoff
- Equal payoffs at T imply equal prices now for European options, so $C_0 = \Delta S_0 - B$

Binomial option pricing: a numerical example

Question 22

Consider the payoff to a long European call option on an underlying stock with share price $S = \$50$, where the share price follows a binomial distribution with $u = 1.2$, $d = 0.8$. Assume an exercise price $X = 50$, expiry date $T = 0.5$, and bond return $R = 0.2$. What Δ and B would replicate the call?

A. $\Delta = 0.5, B = 18.097$

B. $\Delta = 0.5, B = 19.087$

Remarks:

- Buy $\Delta = (C_u - C_d)/\left[S_0(u-d)\right]$ shares, so here buy $\Delta = 10/(60-40)$
- Sell $B = (dC_u - uC_d)/\left[(u-d)e^{RT}\right]$ bonds, so here sell $B = 8/\left[0.4e^{0.1}\right]$
- Using this allocation, the call value must be $C = \Delta S_0 - B = 0.5 \times 50 - 18.097 = 6.903$

Binomial option pricing: a numerical example

Question 22

Consider the payoff to a long European call option on an underlying stock with share price $S = \$50$, where the share price follows a binomial distribution with $u = 1.2$, $d = 0.8$. Assume an exercise price $X = 50$, expiry date $T = 0.5$, and bond return $R = 0.2$. What Δ and B would replicate the call?

A. $\boldsymbol{\Delta = 0.5, B = 18.097}$

B. $\Delta = 0.5, B = 19.087$

Remarks:

- Buy $\Delta = (C_u - C_d)/\left[S_0(u - d)\right]$ shares, so here buy $\Delta = 10/(60 - 40)$
- Sell $B = (dC_u - uC_d)/\left[(u - d)e^{RT}\right]$ bonds, so here sell $B = 8/\left[0.4e^{0.1}\right]$
- Using this allocation, the call value must be $C = \Delta S_0 - B = 0.5 \times 50 - 18.097 = 6.903$

Binomial option pricing: a numerical example

Solution 22

Up and down values of the underlying and the European call option are

$$S_u = uS_0 = 1.2 \times 50 = 60 \quad \Rightarrow \quad C_u = \max(S_u - X, 0) = \max(60 - 50, 0) = 10\,,$$
$$S_d = dS_0 = 0.8 \times 50 = 40 \quad \Rightarrow \quad C_d = \max(S_d - X, 0) = \max(40 - 50, 0) = 0\,.$$

Solutions for replicating portfolio holdings of Δ stocks and B bonds are then

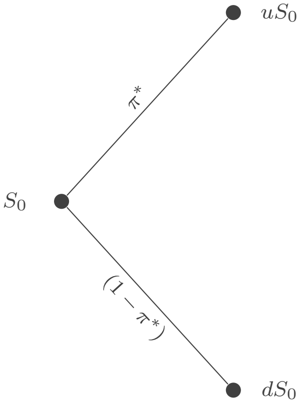
$$\Delta = \frac{C_u - C_d}{S_0(u-d)} = \frac{10}{50(1.2-0.8)} = 10/20 = 0.5\,,$$

$$B = \frac{dC_u - uC_d}{(u-d)e^{RT}} = \frac{0.8 \times 10 - 1.2 \times 0}{(1.2-0.8)e^{0.2 \times 0.5}} = \frac{8}{0.4e^{0.1}} \approx 18.097\,.$$

Lecture 16: Options I
Overview of Topics

Reading: Hillier et al. (2016, Ch 22), Bodie et al. (2014, Ch 21), Hull (2015, Ch 10, 11, 13)

Defining risk-neutral probabilities



- Risk-neutral probabilities are defined by the following equation:

$$\frac{\mathrm{E}[S_T]}{S_0} = \frac{\pi^* u S_0 + (1-\pi^*) d S_0}{S_0} = e^{RT}$$

- The π^* and $1-\pi^*$ are risk-neutral probabilities for price jumps from 0 to T
- Before, we priced a call option using a no-arbitrage condition
- Next, we'll price a call option using these risk-neutral probabilities

Pricing the binomial option using risk-neutral probabilities

$$C_0 = e^{-RT}\big[\pi^* C_u + (1-\pi^*)C_d\big], \quad \text{where } \pi^* = \frac{e^{RT}-d}{u-d}$$

Remarks:

- The call value C_0 equals expected present value of the future payoff C_u or C_d
- From above, the risk-neutral probability is $\pi^* = (e^{0.1} - 0.8)/(1.2 - 0.8) = 0.763$
- The call option value is then $C_0 = e^{-0.1}(0.763 \times 10 + 0.237 \times 0) = 6.903$
- This is the same as valuing the option using the replicating portfolio

Lecture 16: Options I

Revision Checklist

- ☐ Option pricing before expiry
- ☐ Bounds on option prices
- ☐ Binomial option pricing
- ☐ Risk-neutral probabilities

Reading: Hillier et al. (2016, Ch 22), Bodie et al. (2014, Ch 21), Hull (2015, Ch 10, 11, 13)

Intermediate Finance
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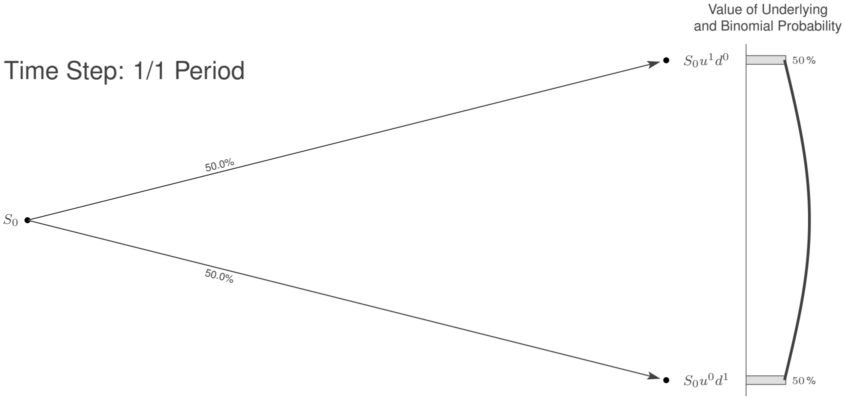
Lecture 17: Options II
Overview of Topics

Reading: Hillier et al. (2016, Ch 22), Bodie et al. (2014, Ch 20 & 21), Hull (2015, Ch 10, 11, & 13)

From binomial option pricing to Black Scholes

Length of Time Step: 1/1 Period

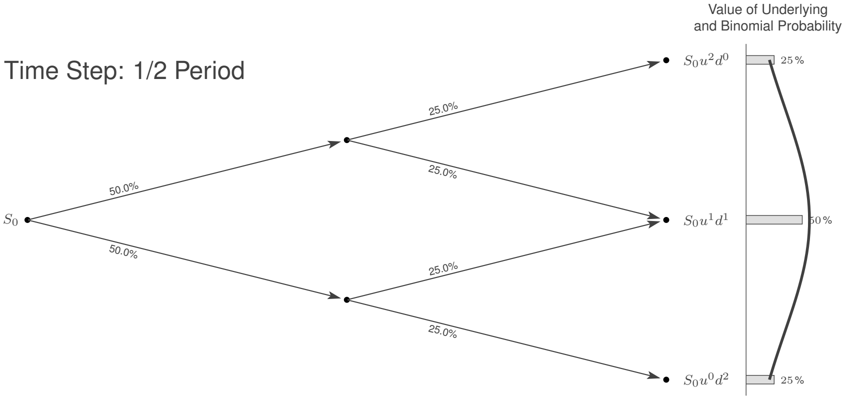


Remarks:

- Initial underlying price S_0 has equal up and down probabilities with ever-finer time steps
- As time steps per period increase, underlying price approaches the normal distribution

From binomial option pricing to Black Scholes

Length of Time Step: 1/2 Period



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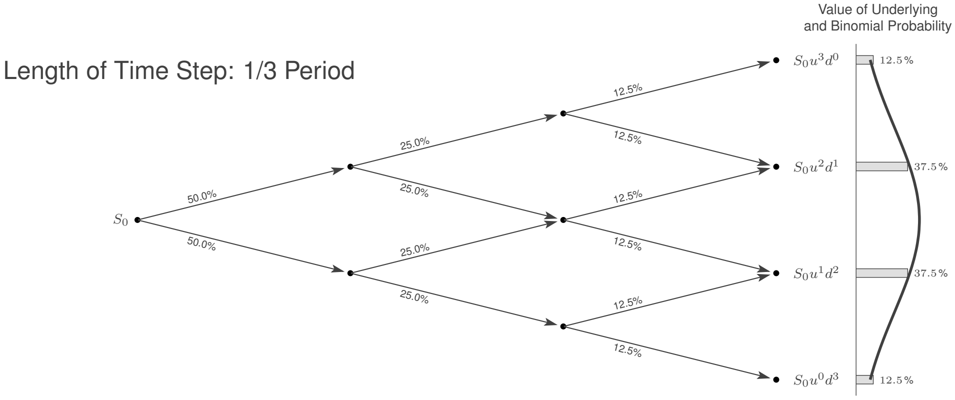
From binomial option pricing to Black Scholes



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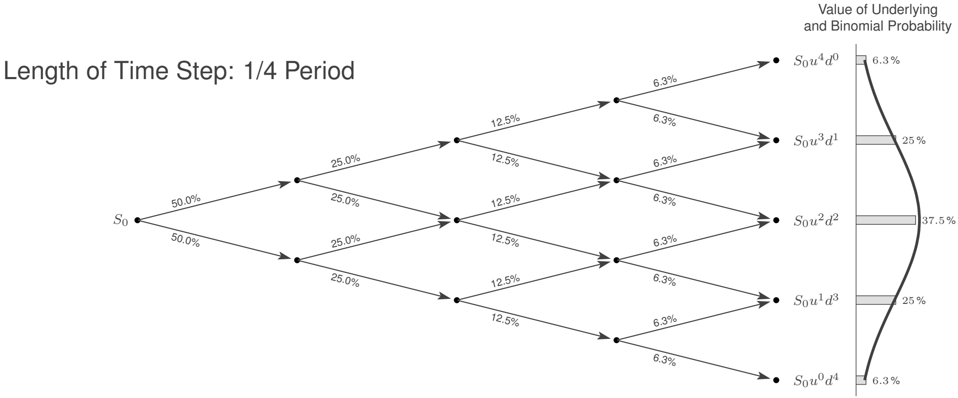
From binomial option pricing to Black Scholes



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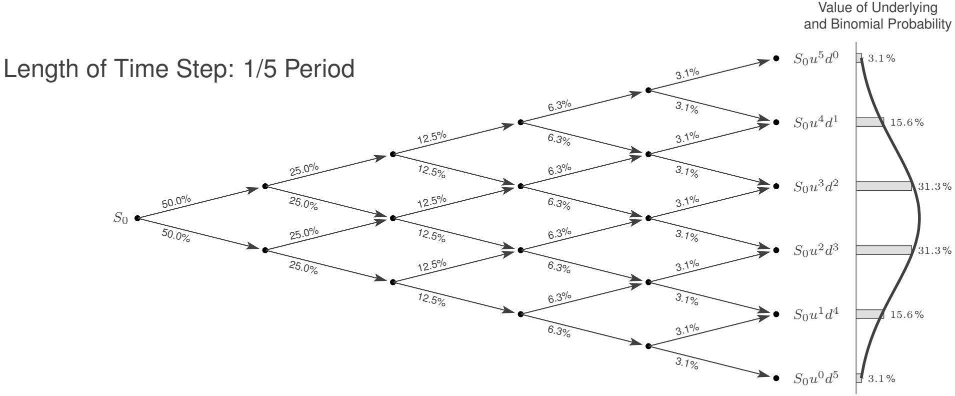
From binomial option pricing to Black Scholes



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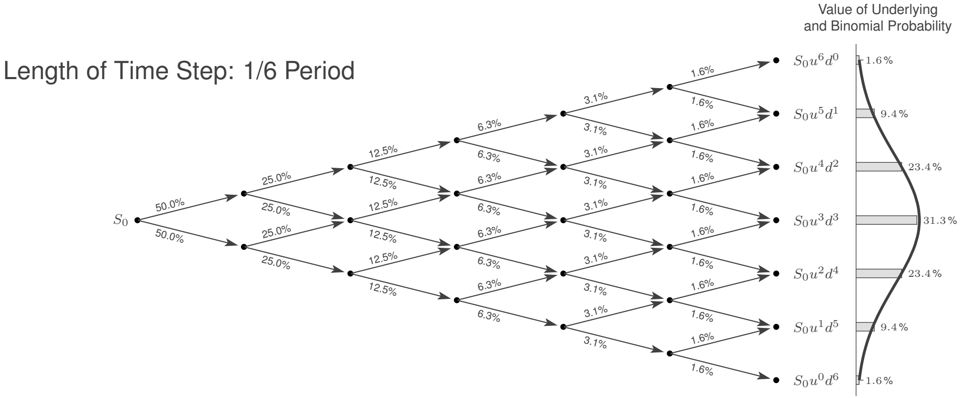
From binomial option pricing to Black Scholes



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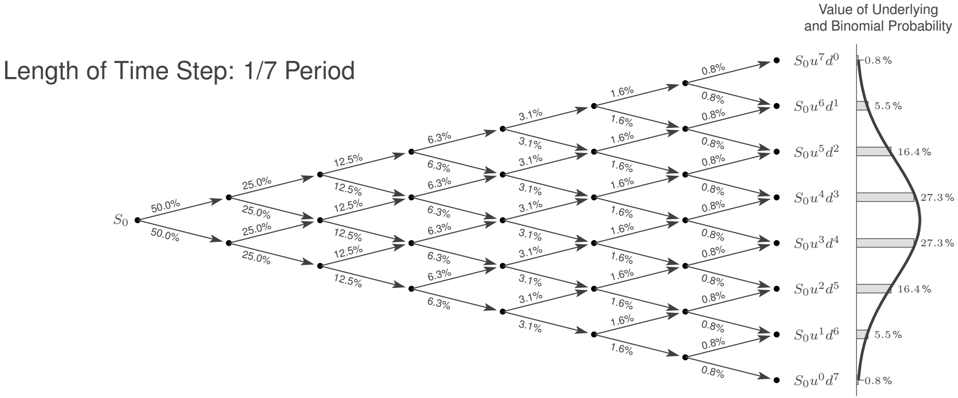
From binomial option pricing to Black Scholes



Remarks:

- Initial underlying price S_0 has equal up and down probabilities with ever-finer time steps
- As time steps per period increase, underlying price approaches the normal distribution

Black Scholes option pricing formulae for calls (C_t) and puts (P_t)

$$C_t = S_t N(d_1) - PV(X) N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_t}{PV(X)}\right)}{\sigma\sqrt{(T-t)}} + \frac{1}{2}\sigma\sqrt{(T-t)}$$

$$P_t = PV(X) N(-d_2) - S_t N(-d_1)$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

Notation:

S_t	underlying spot price at time t
$N(\cdot)$	cumulative probability distribution function for a std normal random variable
$PV(\cdot)$	present value operator, discounting at risk-free rate
X	exercise price for the option
$T-t$	time to expiry for the option
σ	standard deviation of annualized continuously compounded stock return

Black Scholes call price: similar to the binomial call price

Black Scholes Call Price: $C = SN(d_1) - PV(X)N(d_2)$

Binomial Call Price: $C = S\Delta - B$

Remarks:

- Note the similarity between replicating portfolios in these two pricing models
- Black Scholes replicating portfolio: long $N(d_1)$ stocks and short $PV(X)N(d_2)$ bonds
- Binomial replicating portfolio: long Δ stocks and short B bonds
- $N(d_1)$ matches Δ of the binomial option price, $PV(X)N(d_2)$ matches B

Interpretation of the Black Scholes formulae

- Black-Scholes formula implies that a long position in a European-style call option is equivalent to a long position in a replicating portfolio that is simultaneously:
  - long $N(d_1)$ shares of underlying asset S
  - short $N(d_2)$ units of riskless bond $PV(X)$ that pays X at expiry date
- Value of call option and value of replicating portfolio change by same amount in response to small change in value S of underlying asset
  - call option delta $\Delta = N(d_1)$ measures call's sensitivity to small changes in S
  - if S increases by $+1$, C increases by $\Delta = N(d_1)$
- Compare and contrast with hedging portfolio that is simultaneously
  - long 1 call option
  - short $\Delta = N(d_1)$ units of underlying asset
- The term $N(d_2)$ is a risk-neutral probability of exercising the European call at expiry

Lecture 17: Options II
Overview of Topics

Reading: Hillier et al. (2016, Ch 22), Bodie et al. (2014, Ch 20 & 21), Hull (2015, Ch 10, 11, & 13)

Black-Scholes model: a numerical example

Question 23

Shares in a non-dividend-paying stock currently trade at $S = 100$, with annualized volatility of 27.8%. The continuously-compounded risk-free rate equals 6% per annum. What is the value of a 6-month European-style call option with exercise price $X = 105$?

A. 6.99

B. 7.01

Remarks:

- Solve this problem in a few steps using the Black Scholes formulae given above
- First, compute $PV(X)$, then $\sigma\sqrt{T-t}$, then d_1 and d_2, then look up $N(d_1)$ and $N(d_1)$

Black-Scholes model: a numerical example

Question 23

Shares in a non-dividend-paying stock currently trade at $S = 100$, with annualized volatility of 27.8%. The continuously-compounded risk-free rate equals 6% per annum. What is the value of a 6-month European-style call option with exercise price $X = 105$?

A. **6.99**

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Remarks:

- Solve this problem in a few steps using the Black Scholes formulae given above
- First, compute $PV(X)$, then $\sigma\sqrt{T-t}$, then d_1 and d_2, then look up $N(d_1)$ and $N(d_1)$

Black-Scholes model: a numerical example

Solution 23

- Calculate $PV(X)$. Deduce 'moneyness' of option, $S/PV(X)$:
  $PV(X) = 105e^{-0.06*0.5} = 101.9 \quad \Rightarrow \quad \frac{S}{PV(X)} = 0.981$
- Calculate volatility of returns on underlying over remaining term: $\sigma\sqrt{(T-t)}$:
  $\sigma\sqrt{(T-t)} = 0.278 \times \sqrt{(}0.5) = 0.197$
- Calculate d_1 and d_2:
  $d_1 = \frac{ln(0.981)}{0.197} + \frac{1}{2} \times 0.197 = 0.0027$, $d_2 = 0.0027 - 0.197 = -0.194$
- Using linear interpolation, look up $N(d_1)$ and $N(d_1)$:
  $N(d_1) = 0.501 \; N(d_2) = 0.423$
- Compute Black-Scholes European call value $C = SN(d_1) - PV(X)N(d_2)$:
  $C = 100 \times 0.501 - 101.9 \times 0.423 = 6.99$

Lecture 17: Options II
Overview of Topics

Reading: Hillier et al. (2016, Ch 22), Bodie et al. (2014, Ch 20 & 21), Hull (2015, Ch 10, 11, & 13)

The Greeks: partial derivatives of option prices with respect to:

S	spot price
T	time to expiry
σ	volatility
R	riskless rate

Remarks:

- Mathematically, the Greeks are partial derivatives, e.g. $\partial C / \partial S = \Delta$, i.e. "Delta"
- We study how each variable effects the option price, holding other variables constant
- The effects can differ depending on option type: American versus European options
- The effect of changes in exercise price X should be clear ...

The Greeks: European Options

Effect on the Value of a European:			Call Option	Put Option
$\Delta = \partial\Pi/\partial S$	Delta:	increase current spot price	$+$	$-$
$\Theta = \partial\Pi/\partial T$	Theta:	increase time to expiry	$?$	$?$
$V = \partial\Pi/\partial\sigma$	Vega:	increase volatility	$+$	$+$
$\rho = \partial\Pi/\partial R$	Rho:	increase risk-free rate	$+$	$-$

Remarks:

- The variable Π represents an option value, i.e. $\Pi = C$ for calls, $\Pi = P$ for puts
- Increase in underlying price S raises value of call, reduces value of put
- Time to expiry affects volatility $\sigma\sqrt{(T-t)}\uparrow$ and present value $PV(X)\downarrow$
- Increase in volatility σ raises chance of moneyness and thus increases value
- Increase in risk-free rate lowers $PV(X)$, affects lower bounds of call and put

The Greeks: American Options

Effect on the Value of an American:			Call Option	Put Option
$\Delta = \partial\Pi/\partial S$	Delta:	increase current spot price	$+$	$-$
$\Theta = \partial\Pi/\partial T$	Theta:	increase time to expiry	$+$	$+$
$V = \partial\Pi/\partial\sigma$	Vega:	increase volatility	$+$	$+$
$\rho = \partial\Pi/\partial R$	Rho:	increase risk-free rate	$+$	$-$

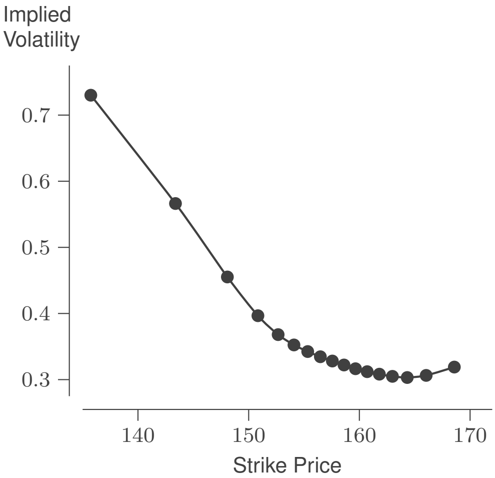
Remarks:

- Assuming a non-dividend yielding asset, American call options are equivalent to European call options
- American puts have payoff $\max(0, X - S_t)$ at expiry; more time to expiry increases value unambiguously

Lecture 17: Options II
Overview of Topics

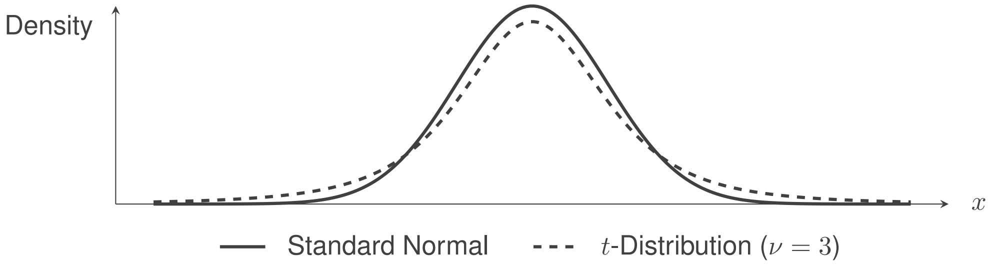
Reading: Hillier et al. (2016, Ch 22), Bodie et al. (2014, Ch 20 & 21), Hull (2015, Ch 10, 11, & 13)

Limitation 1: Constant volatility versus Implied volatility



- The figure shows the implied volatility smile for calls on APPL stock as of Aug 31 2022
- The Black Scholes formula assumes we know the parameters $S, X, T-t, \sigma, R$
- In practice, we observe the option price and 4 of the 5 parameters: $S, X, T-t, R$
- Volatility of the spot price is unobserved, but implied from observed option prices
- Implied volatility is constant in the Black Scholes formula, but not in the data
- In the data, implied volatility and strike price have a u-shaped relationship
- For example, deep in- or out-of-money calls have higher implied volatility

Limitation 2: Normal distribution



Remarks:

- The Black Scholes formula assumes stock returns follow a normal distribution
- The true distribution of stock returns has fatter tails than a normal distribution
- Investors pay higher premia for option insurance when price movements are larger

Cumulative Standard Normal Distribution Probability Tables

Z	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817

Second decimal place of Z										Z
0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00	Z
0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228	−2.0
0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287	−1.9
0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359	−1.8
0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446	−1.7
0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548	−1.6
0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668	−1.5
0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808	−1.4
0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968	−1.3
0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151	−1.2
0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357	−1.1
0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587	−1.0
0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841	−0.9
0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119	−0.8
0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420	−0.7
0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743	−0.6
0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085	−0.5
0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446	−0.4
0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821	−0.3
0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207	−0.2
0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602	−0.1
0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000	−0.0

Lecture 17: Options II

Revision Checklist

- ☐ Black-Scholes formula and interpretation
- ☐ Black-Scholes numerical example
- ☐ The Greeks and option value
- ☐ Limitations of Black-Scholes

Reading: Hillier et al. (2016, Ch 22), Bodie et al. (2014, Ch 20 & 21), Hull (2015, Ch 10, 11, & 13)

Revision Checklist

Lecture 01: Math Refresher (Optional Self-Study)
- ☐ Mathematical Prerequisites
- ☐ Pre-Calculus Refresher
- ☐ Calculus Refresher
- ☐ Statistics Refresher

Lecture 02: Investment Under Certainty
- ☐ Intertemporal utility function
- ☐ Intertemporal budget constraint
- ☐ Capital investment and Fisher separation

Lecture 03: Risk and Expected Return
- ☐ Random variables
- ☐ Discrete Random Variables: Mean and Variance
- ☐ Discrete Random Variables: Comovement
- ☐ Discrete Random Variables: Numerical Examples
- ☐ Continuous Random Variables

Lecture 04: Risk Aversion and Expected Utility I
- ☐ Risk and uncertainty
- ☐ Utility and Risk Aversion
- ☐ Expected wealth and utility

Lecture 05: Risk Aversion and Expected Utility II
- ☐ Certainty equivalent
- ☐ Markowitz risk premium
- ☐ Arrow-Pratt Approximation

Lecture 06: Optimal Portfolio Selection I
- ☐ Portfolios with N assets
- ☐ Expectation and variance of portfolio returns
- ☐ Naive diversification
- ☐ Two-Assets with correlated returns

Lecture 07: Optimal Portfolio Selection II
- ☐ Minimum variance portfolio
- ☐ Capital allocation line
- ☐ Finding the tangency portfolio
- ☐ Lending and borrowing portfolios

Lecture 08: Capital Asset Pricing Model I
- ☐ Comovement with portfolio returns
- ☐ Derivation of beta
- ☐ Efficient frontier with N assets

Lecture 09: Capital Asset Pricing Model II
- ☐ Assumptions of the capital asset pricing model
- ☐ Deriving the capital asset pricing model
- ☐ Securities market line
- ☐ Empirical failures of the capital asset pricing model

Lecture 10: Market Efficiency I
- ☐ Forms of Market Efficiency
- ☐ Two EMH Testing Strategies
- ☐ Summary

Lecture 11: Market Efficiency II
- ☐ Event Studies and Abnormal Returns
- ☐ Empirical Evidence on Market Reactions
- ☐ Market Efficiency and Professional Investing

Lecture 12: Bond Pricing I
- ☐ Coupon and zero-coupon bonds
- ☐ The yield curve three ways
- ☐ Two-period forward rate
- ☐ Multi-period forward rate

Lecture 13: Bond Pricing II
- ☐ Yield curve arbitrage
- ☐ Bootstrapping the yield curve
- ☐ Three theories of the yield curve

Lecture 14: Forwards and Futures I
- ☐ Forwards and futures
- ☐ Payoff diagrams
- ☐ Spot-forward parity
- ☐ Forward price arbitrage

Lecture 15: Forwards and Futures II
- ☐ Short and long calls and puts
- ☐ Constructing forwards from options
- ☐ Put-call parity
- ☐ Put-call arbitrage

Lecture 16: Options I
- ☐ Option pricing before expiry
- ☐ Bounds on option prices
- ☐ Binomial option pricing
- ☐ Risk-neutral probabilities

Lecture 17: Options II
- ☐ Black-Scholes formula and interpretation
- ☐ Black-Scholes numerical example
- ☐ The Greeks and option value
- ☐ Limitations of Black-Scholes

References

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