On Aggregate Fluctuations, Systemic Risk, and the Covariance of Firm-Level Activity

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On Aggregate Fluctuations, Systemic Risk, and the Covariance of Firm-Level Activity

Where do fluctuations in aggregate economic activity originate?

Shocks to all firms? Oil, credit conditions, monetary and fiscal policy...

- Empirically, leaves some aggregate variance unexplained

Shocks to huge firms? Idiosyncratic shocks, fat-tailed firm size distribution...

- Trouble explaining productivity comovement and firm-level risk

Shocks to networked firms? Idiosyncratic shocks, input-output networks...

- Trouble explaining productivity comovement, maybe also firm-level risk

Where do fluctuations in aggregate economic activity originate? From endogenous co-movement in firm-level productivity

Motivating evidence:

- Three stylized facts on firm-level co-movement using Compustat data
- Based on simple aggregate variance decomposition

Theoretical framework:

- Tractable DSGE production economy with multi-product, multi-technology firms
- Endogenous prob distributions and covariance for firm and aggregate productivity

Regression analysis:

- Plausibility check on key predictions of the model
- Tentative empirical support for TFP comovement hypothesis

My work relates to three strands of literature in macro and finance

Origins of Aggregate Fluctuations:

- Herskovic et al. (2017), Carvalho and Gabaix (2013), Acemoglu et al. (2012), Gabaix (2011)
- Idiosyncratic shocks to large or “hub” firms generate aggregate fluctuations
- Me: Firm-level productivity comovement drives aggregate fluctuations

General Equilibrium Asset Pricing:

- Clementi and Palazzo (2018), Zhang (2017), İmrohoroğlu and Tuzel (2014)
- Idiosyncratic shocks and adjustment costs drive differences in risk
- Me: Differences in exposure to common shocks drive differences in risk

Endogenous Risk:

- Romer (2016), Carvalho and Grassi (2019), Stiglitz (2011), Danielsson and Shin (2003)
- Agents should choose their risks, chosen risks should *become* systemic
- Me: Firms choose their risks, making firm-level uncertainty endogenous

On Aggregate Fluctuations, Systemic Risk, and the Covariance of Firm-Level Activity

A simple aggregate variance decomposition

For a firm variable x_ω and an aggregate variable $X = \sum_{\omega \in \Omega} s_\omega x_\omega$:

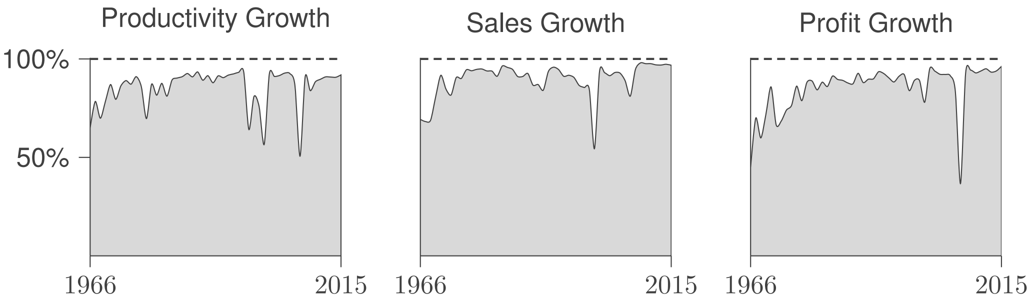
$$\text{Var}(X) = \sum_{\Omega} s_\omega^2 \text{Var}(x_\omega) + \sum_{\Omega} \sum_{\Omega \setminus \{\omega\}} s_\omega s_{\omega'} \text{Cov}(x_\omega, x_{\omega'}) = \sum_{\Omega} s_\omega \text{Cov}(x_\omega, X)$$

Stylized facts:

1. Aggregate variance comes mostly from pairwise covariances
2. High-productivity firms contribute more to aggregate variance
3. High-productivity firms contribute less per dollar of market value

TFP Details

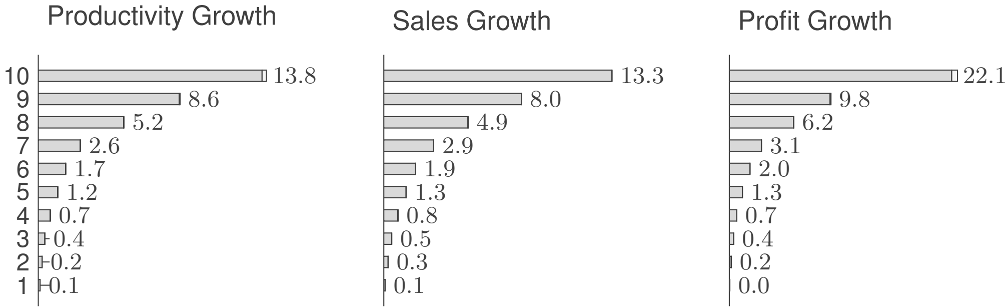
1. Aggregate variance comes mostly from pairwise covariances



Remarks:

- Weighted sum of pairwise covariances divided by aggregate variance
- Variances and covariances in 6-year rolling windows, Compustat 1966–2015

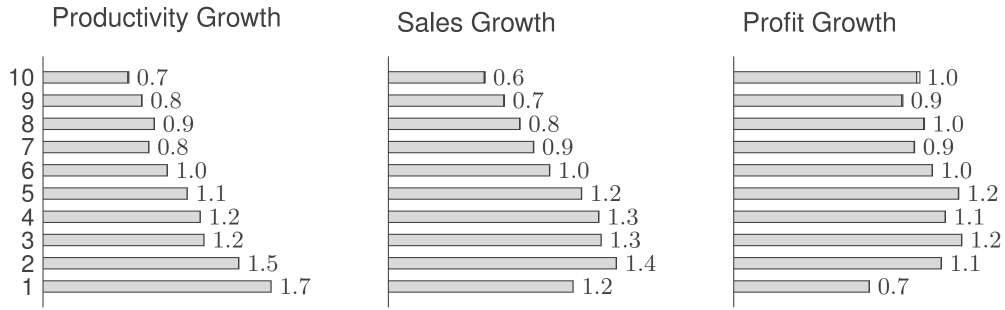
2. High-productivity firms contribute *most* to aggregate variance



Remarks:

- Weighted covariance between firm and aggregate, Compustat 1966–2015
- Decile median relative to cross-sectional median, averaged over 6-year rolling windows

3. High-productivity firms contribute *least* per dollar of market value



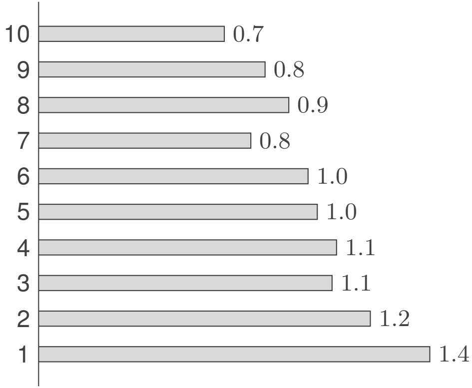
Remarks:

- Weighted covariance between firm and aggregate, relative to firm market value
- Decile median relative to cross-sectional median, averaged over 6-year rolling windows

2PCH

Bottom line: Existing theories don't capture all the evidence

Excess Returns*



*As reported in İmrohoroğlu and Tuzel (2014)

Remarks:

- Aggregate fluctuations come mostly from pairwise covariances, not individual variances
- High-productivity firms contribute *most* to aggregate fluctuations
- High-productivity firms contribute *least* per dollar of market value
- This may help explain high-productivity firms' lower excess returns

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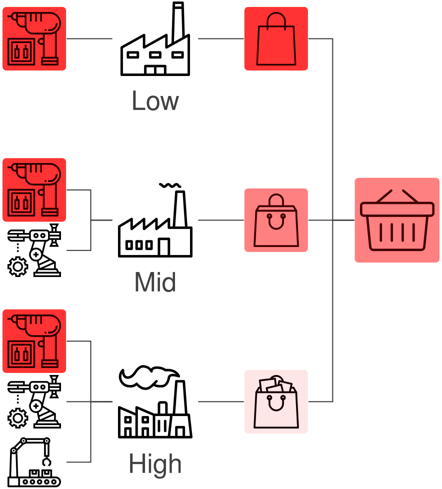
Common technology shocks drive covariance patterns



1. Heterogeneous firms
   - Some firms managed well
   - Others managed poorly
2. Heterogeneous technology
   - Some technologies cheap
   - Others expensive
3. Differentiated goods
   - Goods are firm-tech specific
   - Household consumes a basket
4. Common shocks
   - Cheap technologies are systemic
   - Low-prod firms are more exposed

Firms earn profit using technology to produce differentiated goods

Firms indexed ω produce multiple varieties v, each with a different technology:

$$y_t(v,\omega) = z(\omega) z_t(v) \big[k_t(v,\omega)\big]^{\alpha} \big[l_t(v,\omega)\big]^{1-\alpha}, \tag{1}$$

where $z_t(v)$ is stochastic, and $z(\omega)$ is not. Gross profit from each variety is:

$$\pi_t(v,\omega) = p_t(v,\omega) y_t(v,\omega) - r_t k_t(v,\omega) - w_t l_t(v,\omega). \tag{2}$$

Remarks:

- Two productivities: firm-specific non-random $z(\omega)$, technology-specific random $z_t(v)$
- Firms also pay a period fixed cost $f_t(v)$ for each technology

Firms choose inputs and prices to maximize profit each period

Firms face downward-sloping demand for each variety:

$$c_t(v,\omega) = \left[p_t(v,\omega)\right]^{-\theta} C_t. \tag{3}$$

Firms choose prices and production factors to maximize profit:

$$\begin{aligned} \max_{\left\{\substack{k_t(v,\omega), \\ l_t(v,\omega), \\ p_t(v,\omega)}\right\}_{v\in\mathcal{V}(\omega)}} \quad & \Pi_t(\omega) = \int_{\mathcal{V}(\omega)} \pi_t(v,\omega)\lambda(dv) \\ \text{s.t.} \quad & \text{(1) and (31)} \quad \forall\, v \in \mathcal{V}(\omega). \end{aligned} \tag{4}$$

Firms must also choose their technology sets $\mathcal{V}(\omega) \subseteq \mathcal{V} = [\underline{v}, \infty)$.

First-Order Conditions

Firms choose next-period technologies to maximize expected profit

Firms operate any technology v with positive expected net present value:

$$\mathrm{E}_t\left[m_{t,t+1}(\pi_{t+1}(v,\omega) - f_{t+1}(v))\right] > 0. \tag{5}$$

Firms pay a fixed cost to operate technology v each period:

$$f_{t+1}(v) = \frac{Y_{t+1}}{\mu} v^{\gamma}. \tag{6}$$

Remarks:

- No sunk cost to adopt or abandon a technology
- Fixed cost assumed to rise and fall with aggregate production

Household maximizes expected discounted lifetime utility

The household maximizes utility:

$$\max_{\left\{\substack{C_s,\\ K_{s+1},\\ S_{s+1}(\omega)}\right\}_{\substack{s\in\mathcal{T}_t,\\ \omega\in\Omega}}} \quad U_t = \mathrm{E}\left[\sum_{s=t}^{\infty} \beta^{s-t} \ln(C_s)\right] \tag{7}$$

$$\text{s.t.} \quad \text{(9) and (10)} \quad \forall\, s \geq t,$$

where the consumption basket is

$$C_t = \left[\int_{\Omega}\int_{\mathcal{V}(\omega)} \left[c_t(v,\omega)\right]^{\frac{\theta-1}{\theta}} \lambda(dv d\omega)\right]^{\frac{\theta}{\theta-1}}. \tag{8}$$

First-Order Conditions

Household buys capital and consumption with its income

Budget constraint:

$$w_t L + r_t K_t + \int_\Omega \left[\Pi_t(\omega) - F_t(\omega)\right] S_t(\omega)\lambda(d\omega)$$
$$= C_t + I_t + \int_\Omega V_t(\omega)\left[S_{t+1}(\omega) - S_t(\omega)\right]\lambda(d\omega). \tag{9}$$

Capital accumulation:

$$K_{t+1} = I_t + (1-\delta)K_t. \tag{10}$$

Remarks:

- Capital is produced in a separate sector to simplify goods market clearing
- I omit the primitives related to capital production here

Households differentiate goods by producer and technology I

The household's second stage problem:

$$\max_{\{c_t(v,\omega)\}_{v\in\mathcal{V},\ \omega\in\Omega}} \quad C_t = \left[\int_\Omega \int_{\mathcal{V}(\omega)} [c_t(v,\omega)]^{\frac{\theta-1}{\theta}} \lambda(dvd\omega) \right]^{\frac{\theta}{\theta-1}} \tag{11}$$

$$\text{s.t.} \quad 1 = \int_\Omega \int_{\mathcal{V}(\omega)} p_t(v,\omega) c_t(v,\omega) \lambda(dvd\omega)$$

Remarks:

- Household optimally allocates resources among differentiated goods
- Goods are differentiated by producer and by production technology

I’ve complicated the standard model—now what does it buy me?

What’s new here?

- Many technologies exist, each a distinct source of randomness (more uncertainty)
- Some technologies are cheap to operate, others are expensive (more heterogeneity)
- Each technology allows a firm to make a new differentiated good (additional margin)

What does this set-up buy me?

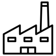
- A microeconomic explanation for aggregate fluctuations
- A microeconomic explanation for firm-level systematic risk
- All fluctuations and risk are endogenous to firm-level decisions!

Can’t come cheap, right? Wrong — no loss of tractability b/c model aggregates nicely

Proposition 1: Aggregation lets us view the model at three levels

Economy-Wide: $$Y_t = Z_t\big[K_t\big]^{\alpha}\big[L\big]^{1-\alpha}$$

Firm-Level: $$Y_t(\omega) = Z_t(\omega)\big[K_t(\omega)\big]^{\alpha}\big[L_t(\omega)\big]^{1-\alpha}$$



Product-Level: $$y_t(v,\omega) = z(\omega)z_t(v)\big[k_t(v,\omega)\big]^{\alpha}\big[l_t(v,\omega)\big]^{1-\alpha}$$

Remarks:

- Outputs Y_t and $Y_t(\omega)$ are each Dixit-Stiglitz aggregates of lower-level aggregates
- Aggregate factor demands and profit are also written in terms of Z_t and $Z_t(\omega)$

Proposition 1 (Aggregation)

A productivity aggregate over technologies summarizes all of the technological heterogeneity within an individual firm ω:

$$Z_t(\omega) = \left[\int_{\mathcal{V}(\omega)} \left[z(\omega) z_t(v) \right]^{\theta-1} \lambda(dv) \right]^{\frac{1}{\theta-1}}. \tag{12}$$

A productivity aggregate over firms summarizes all of the firm-specific and technological heterogeneity within the consumption goods sector:

$$Z_t = \left[\int_{\Omega} Z_t(\omega)^{\theta-1} \lambda(d\omega) \right]^{\frac{1}{\theta-1}}. \tag{13}$$

Aggregate factor demands, production, and profit can be written in terms of aggregate productivities and variables that either do not vary across firms, in the case of firm aggregates, or do not vary across firms or technologies, in the case of economy-wide aggregates.

Derivation

Integral sums of random variables? How is uncertainty preserved? I

A simple construction inspired by Al-Najjar (1995) preserves risk in the continuum:

$$z_t(v)^{\theta-1} := \epsilon_{t,\lceil v \rceil} \quad \forall\, v \in \mathcal{V},$$

with $\mathcal{E} = \{\epsilon_{t,1}, \epsilon_{t,2}, \dots\}$ a *countable* set of random variables, and with:

$$\mathrm{E}\left[\epsilon_{t,n}\right] = \mu_\epsilon \quad \forall\, n \in \mathbb{N},$$

$$\mathrm{Var}\left(\epsilon_{t,n}\right) = \sigma_\epsilon^2 \quad \forall\, n \in \mathbb{N},$$

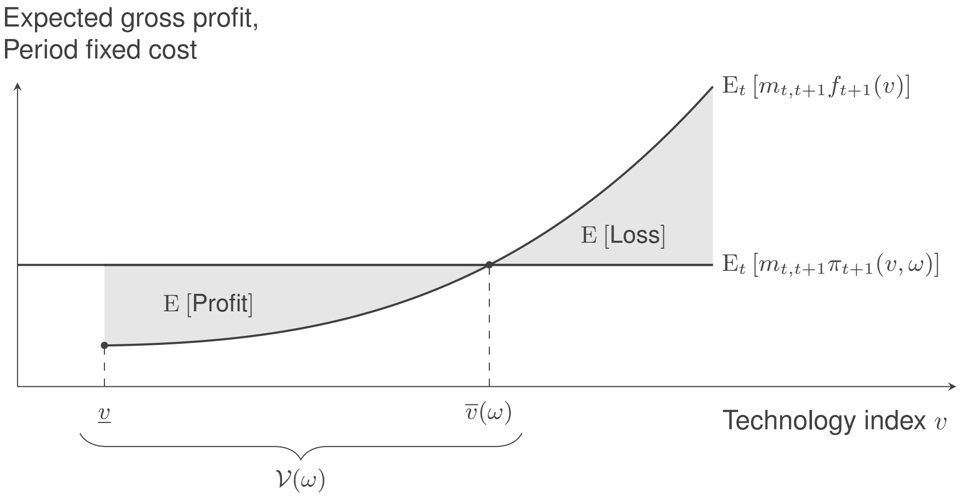
$$\mathrm{Cov}\left(\epsilon_{t,n}, \epsilon_{t,m}\right) = 0 \quad \forall\, n \neq m, \quad n, m \in \mathbb{N},$$

$$\mathrm{Cov}\left(\epsilon_{s,n}, \epsilon_{t,n}\right) = 0 \quad \forall\, n \in \mathbb{N}, \quad s \neq t \in \mathbb{Z}.$$

Remarks:

- Each ϵ_n is associated with a unit interval of $z_t(v)$'s, hence $z_t(v)$'s are not independent
- Interpret ϵ_n's as fundamental technologies, $z_t(v)$'s as commercial applications

Proposition 2: Technology sets are chosen by firms



Proposition 2 (Technology Sets)

In non-stochastic steady state, any firm ω with productivity $z(\omega) \geq \underline{z}$ chooses technology set $\mathcal{V}(\omega) = \left\{v \in \mathcal{V} : \underline{v} \leq v \leq \overline{v}(\omega)\right\}$, where the endogenous cut-offs $\underline{z}$ and $\overline{v}(\omega)$ are given by:

$$\underline{z} = \left(\frac{\theta}{\mu_\epsilon}\right)^{\frac{1}{\theta-1}} \tag{14}$$

$$\overline{v}(\omega) = \left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\gamma}} z(\omega)^{\frac{\theta-1}{\gamma}}. \tag{15}$$

Firms with $z(\omega) < \underline{z}$ do not produce. Under parameter restrictions, firms ω_1 and ω_2 with productivities $\underline{z} < z(\omega_1) < z(\omega_2)$ choose technology sets such that $\mathcal{V}_t(\omega_1) \subset \mathcal{V}_t(\omega_2)$. The above cut-offs are also first-order approximate to those that obtain in a stochastic environment.

Derivation

Proposition 3 (Endogenous First and Second Moments)

Let technology sets be those that firms choose in non-stochastic steady state. Then the first and second moments of firm-level productivity are given by $\mu(\omega)$ and $\sigma^2(\omega)$, respectively:

$$\mu(\omega) = \mu_\epsilon z(\omega)^{\zeta_{\mu\omega 1}} \left[\left(\frac{z(\omega)}{\underline{z}} \right)^{\zeta_{\mu\omega 2}} - 1 \right], \tag{16}$$

$$\sigma^2(\omega) = \sigma_\epsilon^2 z(\omega)^{\zeta_{\sigma\omega 1}} \left[\left(\frac{z(\omega)}{\underline{z}} \right)^{\zeta_{\sigma\omega 2}} - 1 \right]. \tag{17}$$

The first and second moments of aggregate productivity are given by μ and σ^2, respectively:

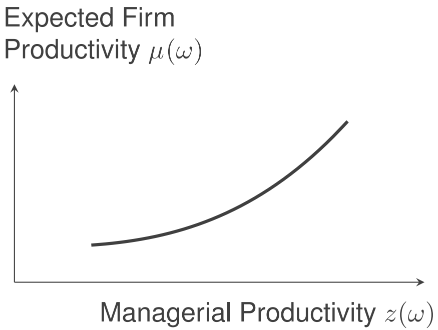
$$\mu = \mu_\epsilon \zeta_{\mu 1} \underline{z}^{\zeta_{\mu 2}}, \tag{18}$$

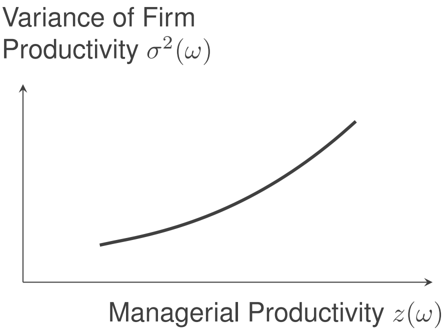
$$\sigma^2 = \sigma_\epsilon^2 \zeta_{\sigma 1} \underline{z}^{\zeta_{\sigma 2}}. \tag{19}$$

Under parameter restrictions, the first and second moments of all productivity aggregates are positive and finite. For firms ω_1 and ω_2 with $z(\omega_1) < z(\omega_2)$, we have $\mu_t(\omega_1) < \mu_t(\omega_2)$ and $\sigma_t^2(\omega_1) < \sigma_t^2(\omega_2)$.

FNZSM Intuition Derivation

Well-managed firms have higher and more volatile productivity

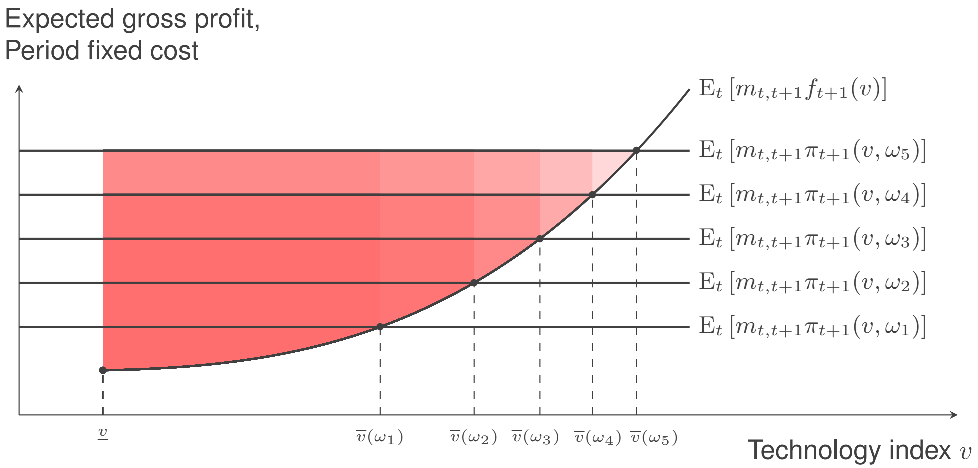




Remarks:

- Ad-hoc parameterization $\beta = 0.99$, $\delta = 0.1$, $\alpha = 0.25$, $\theta = 3.8$, $\kappa = 3.4$, $\gamma = 6$
- "Managerial productivity" is shorthand for firm-specific, non-technological productivity

Proposition 4: Covariance driven by overlapping technology



Proposition 4 (Firm-Aggregate Covariance)

Let technology sets be those that firms choose in the non-stochastic steady state. Then the covariance between firm and aggregate productivity, denoted by $\sigma_{\omega\Omega}(\omega) = \text{Cov}\left(Z_t(\omega)^{\theta-1}, Z_t^{\theta-1}\right)$, is given by

$$\sigma_{\omega\Omega}(\omega) = z(\omega)^{\theta-1}\zeta_{\omega\Omega 1}\left[1 - \left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega 2}}\right] \tag{20}$$

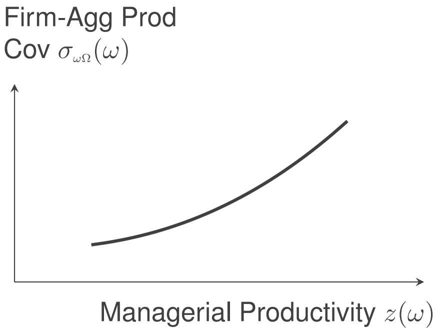
The covariance between firm and aggregate productivity, expressed as a fraction of firm market value, is approximated to a first order by

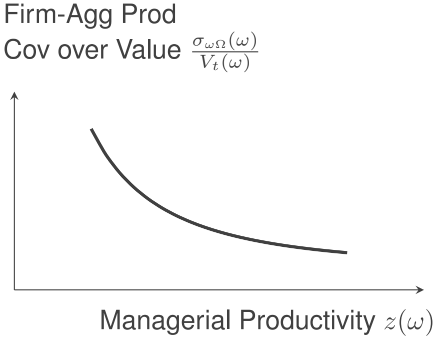
$$\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)} \approx \frac{1}{Y_t}\left(\frac{\zeta_{\omega\Omega 1}\left[1 - \left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega 2}}\right]}{\zeta_{V1}\left(\frac{z(\omega)}{\underline{z}}\right)^{\zeta_{V2}} + \zeta_{V3}\left(\frac{1}{z(\omega)}\right)^{\zeta_{V4}} - \left(\frac{1}{\underline{z}}\right)^{\zeta_{V4}}}\right). \tag{21}$$

Under parameter restrictions, covariance-over-value falls for all $z(\omega)$ above a threshold. The ratio also falls in the level of aggregate output.

Derivation

Well-managed firms covary more w/ aggregate productivity, but less per dollar of market value





Remarks:

- Firm-aggregate productivity covariance is higher for high-productivity firms
- Covariance-over-value lower for high-productivity firms, as in Compustat data

Proposition 5 (Stock Returns)

Let technology sets be those that firms choose in the non-stochastic steady state. Then firm ω's expected excess return is approximated to a second order by

$$\mathrm{E}_t\left[r_{t+1}(\omega) - r_{f,t+1}\right] \approx \zeta_{r1}\frac{\mu(\omega)}{V_t(\omega)} + \zeta_{r2}\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)}, \tag{22}$$

where I define firm ω's return as $r_t(\omega) = \left[V_{t+1}(\omega) + \Pi_{t+1}(\omega) - F_{t+1}(\omega)\right]/V_t(\omega)$, and the risk-free rate as $r_{f,t} = m_{t,t+1}^{-1}$. Under parameter restrictions, expected excess returns decrease in firm productivity $z(\omega)$ for all $z(\omega)$ above a threshold.

Derivation

On Aggregate Fluctuations, Systemic Risk, and the Covariance of Firm-Level Activity

Does TFP help explain patterns in firm-level covariance?

Testable predictions:

1. Ceteris paribus, covariance between firm and aggregate growth rates increases in firm-level total factor productivity.

2. Ceteris paribus, covariance between firm and aggregate growth rates over market value decreases in firm-level total factor productivity.

Compustat: Standard & Poor's data on U.S. public firms I

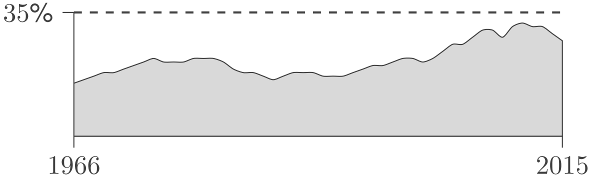
Basic description:

- 50 years: 1966–2015
- about 7,500 firms
- about 20% of yearly GDP
- all industries covered

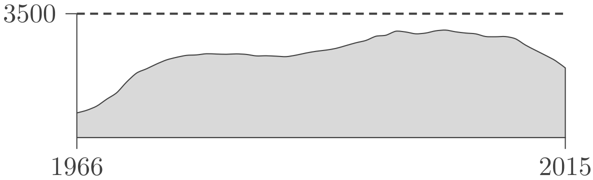
(Dis)advantages:

- Easy to access / replicate
- Not just manufacturing
- No small firms
- No private firms

Compustat Value Added over Real GDP



Yearly Number of Firms in Compustat



Details

Does TFP help explain patterns in firm-level covariance?

$$\frac{s_\omega \operatorname{Cov}(x_\omega, X)}{\text{Base}} = \beta_0 + \beta_1 \left(\substack{\text{Total Factor-} \\ \text{Productivity}_\omega} \right) + \beta_2 \left(\substack{\text{Financial} \\ \text{Strength}_\omega} \right) + \beta_3 \left(\substack{\text{Other} \\ \text{Controls}_{\omega,\Omega}} \right) + \epsilon_\omega,$$

Remarks:

- Base is either aggregate variance $\operatorname{Var}(X)$, or firm market value $V_t(\omega)$
- Variables x_ω and X are productivity, sales, and profit growth
- Estimated dependent variable, so robust standard errors (Lewis and Linzer, 2005)
- Compustat diversification measures don't work (Villalonga, 2004); Census data needed

Regression: $s_\omega \, \mathrm{Cov}\,(x_\omega, X) \,/\, \mathrm{Var}\,(X)$ on explanatory variables

	Growth Rates		
	x, X = TFP	Sales	Profit
Olley-Pakes Total Factor Productivity	0.053^{*}	0.052^{*}	0.171^{***}
	(0.028)	(0.026)	(0.037)
Debt-to-Book Equity	0.006	0.002	0.004
	(0.005)	(0.004)	(0.004)
Quick Ratio	−0.000	−0.001	0.001
	(0.000)	(0.001)	(0.001)
Years in Compustat	0.022^{***}	−0.005	−0.001
	(0.007)	(0.007)	(0.008)
Employment Share	0.124^{***}	0.358^{***}	0.253^{***}
	(0.026)	(0.034)	(0.029)
R-squared	0.576	0.502	0.551

Regression: $s_\omega \operatorname{Cov}(x_\omega, X) / V_t(\omega)$ on explanatory variables

	Growth Rates		
	x, X = TFP	Sales	Profit
Olley-Pakes Total Factor Productivity	-0.059^{***}	-0.015^{**}	-0.033^{***}
	(0.013)	(0.008)	(0.012)
Debt-to-Book Equity	-0.018	0.005	0.003
	(0.027)	(0.013)	(0.010)
Quick Ratio	-0.008^{***}	-0.015^{***}	-0.011^{***}
	(0.002)	(0.004)	(0.002)
Years in Compustat	-0.103^{***}	-0.268^{***}	-0.144^{***}
	(0.007)	(0.008)	(0.007)
Employment Share	-0.000	0.012	0.035^{***}
	(0.006)	(0.010)	(0.008)
R-squared	0.443	0.420	0.368

Robust standard errors in parentheses; * p<0.10, ** p<0.05, *** p<0.01

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Summing up: what's novel, what's useful, why is it important?

Novelty: endogenous productivity comovement

1. firms choose their risks, and the chosen risks become systemic
2. firm-level productivity comovement as driver of aggregate fluctuations

Usefulness: highly tractable model

1. model aggregates, despite technology heterogeneity and choice problem
2. model preserves aggregate risk despite continuum of shocks (Al-Najjar, 1995)

Importance: post-crisis criticism of macro models

- Aggregate fluctuations, systemic risk, two huge post-crisis questions
- If risk is endogenous, policymakers can do more than react—they can preempt!

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Appendix

Profit Comovement

Model derivations

Olley-Pakes Productivity

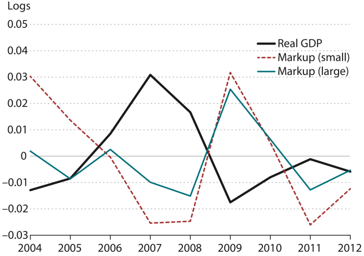
Compustat Dataset

İmrohoroğlu and Tuzel (2014)

References

Can time-varying markups explains patterns in profit comovement? I

Back to evidence

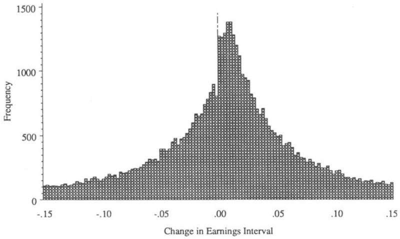


Remarks:

- Evidence from Hong (2018), Bureau van Dijk French manufacturing
- Small firm markups *more* countercyclical, so profits *less* procyclical
- Detrended real GDP; markups estimated following De Loecker et al. (2017)

Can earnings management distorts patterns in profit comovement? I

Back to evidence



Remarks:

- Evidence from Burgstahler and Dichev (1997), Compustat 1976–1994
- Empirical distribution of annual changes in profit over market value
- Small losses much less likely than “normal” to be reported

Appendix

First-Order Conditions I

Back to HH Problem Back to Firm Problem

Optimality conditions for the representative household. The household solves its utility maximization problem in two stages. The two-stage budgeting procedure is possible here because the period utility function $u(C_s)$ depends only on the basket C_t, and C_t is homogeneous of degree one (Gorman, 1959). Consider the first-stage problem in (7). Eliminate constraint (10) by substituting for I_t in (9). Use the method of Lagrangian multipliers to rewrite the objective function as

$$\mathcal{L} = \mathrm{E}\left[\sum_{s=t}^{\infty} \beta^{s-t} u(C_s) - \beta^{s-t}\lambda_s \left(C_s + K_{s+1} + \int_{\omega\in\Omega} V_s(\omega) S_{s+1}(\omega)\lambda(d\omega) \right.\right.$$
$$\left.\left. - w_s L - (1 + r_s - \delta)K_s - \int_{\omega\in\Omega} [V_s(\omega) - \Pi_s(\omega)] S_s(\omega)\lambda(d\omega) \right)\right]. \quad (23)$$

First-Order Conditions II

Back to HH Problem Back to Firm Problem

To get first-order optimality conditions, equate with zero the first derivatives of $\mathcal{L}$ with respect to choice variables C_s, K_{s+1}, $S_{s+1}(\omega)$, and λ_s for arbitrary period s and firm ω. The household's optimal plans for consumption, capital accumulation, and equity shares, respectively, satisfy the following conditions:

$$\mathrm{E}\left[u'(C_s)\right] = \mathrm{E}\left[\lambda_s\right], \tag{24}$$

$$\mathrm{E}\left[\lambda_s\right] = \beta\mathrm{E}\left[\lambda_{s+1}(1 + r_{s+1} - \delta)\right], \tag{25}$$

$$\mathrm{E}\left[\lambda_s V_s(\omega)\right] = \beta\mathrm{E}\left[\lambda_{s+1}(V_{s+1}(\omega) + \Pi_{s+1}(\omega))\right]. \tag{26}$$

The household's stochastic discount factor also derives from these conditions: set $s = t$ and use (24) and (26) to write firm ω's period-t present value as

$$V_t(\omega) = \mathrm{E}_t\left[\left(\beta\frac{u'(C_{t+1})}{u'(C_t)}\right)\left(V_{t+1}(\omega) + \Pi_{t+1}(\omega)\right)\right]. \tag{27}$$

First-Order Conditions III

▸ Back to HH Problem ▸ Back to Firm Problem

The one-period stochastic discount factor is then the first term in the expectation operator: $m_{t,t+1} = \beta u'(C_{t+1})/u'(C_t)$. Iterate (27) via $V_{t+1}(\omega)$ to get the multi-period stochastic discount factor. For any period $s \geq t$, write the latter as

$$\begin{aligned} m_{t,s} &= m_{t,t+1} \cdot m_{t+1,t+2} \cdots m_{s-1,s} \\ &= \beta \frac{u'(C_{t+1})}{u'(C_t)} \cdot \beta \frac{u'(C_{t+2})}{u'(C_{t+1})} \cdots \beta \frac{u'(C_s)}{u'(C_{s-1})} = \beta^{s-t} \frac{u'(C_s)}{u'(C_t)}. \end{aligned} \tag{28}$$

Next, solve the household's second-stage problem of allocating consumption across varieties $c_t(v, \omega)$ within the aggregate basket C_t. Let $P_t(v, \omega)$ be the nominal price of variety $c_t(v, \omega)$, and P_t be the nominal price of the consumption basket C_t. The household takes the optimal amount of aggregate consumption C_t as given by the first-stage problem, and takes nominal prices as given, and maximizes its consumption of varieties for each unit of expenditure $1 := P_t C_t$, by solving eq. (11). Writing the Lagrangian,

$$\mathcal{L} = \left[\int_{\Omega} \int_{\mathcal{V}(\omega)} \left[c_t(v, \omega) \right]^{\frac{\theta - 1}{\theta}} \lambda(dv d\omega) \right]^{\frac{\theta}{\theta - 1}} + \lambda_t \left[1 - \int_{\Omega} \int_{\mathcal{V}(\omega)} P_t(v, \omega) c_t(v, \omega) \lambda(dv d\omega) \right].$$

First-Order Conditions IV

▸ Back to HH Problem ▸ Back to Firm Problem

Taking the first derivative of the Lagrangian with respect to consumption varieties $c_t(v, \omega), c_t(v', \omega')$, and setting equal to zero,

$$C_t^{-1} c_t(v, \omega)^{-\frac{1}{\theta}} = P_t(v, \omega),$$
$$C_t^{-1} c_t(v', \omega')^{-\frac{1}{\theta}} = P_t(v', \omega'),$$

and the ratio of the two optimality conditions yields,

$$\left(\frac{c_t(v, \omega)}{c_t(v', \omega')}\right)^{-\frac{1}{\theta}} = \frac{P_t(v, \omega)}{P_t(v', \omega')}. \tag{29}$$

First-Order Conditions V

▸ Back to HH Problem ▸ Back to Firm Problem

Using eq. (29) in the expenditure constraint in eq. (11),

$$
\begin{aligned}
1 &= \int_{\Omega} \int_{\mathcal{V}(\omega)} P_t(v,\omega) c_t(v,\omega) \lambda(dv d\omega) \\
&= \int_{\Omega} \int_{\mathcal{V}(\omega)} P_t(v,\omega) \left(\frac{P_t(v',\omega')}{P_t(v,\omega)} \right)^{\theta} c_t(v',\omega') \lambda(dv d\omega) \\
&= P_t(v',\omega')^{\theta} c_t(v',\omega') \int_{\Omega} \int_{\mathcal{V}(\omega)} P_t(v,\omega)^{1-\theta} \lambda(dv d\omega).
\end{aligned}
$$

First-Order Conditions VI

▸ Back to HH Problem ▸ Back to Firm Problem

Again using eq. (29), notice that the aggregate consumption basket can be written

$$C_t = \left[\int_{\Omega} \int_{\mathcal{V}(\omega)} \left[c_t(v,\omega) \right]^{\frac{\theta-1}{\theta}} \lambda(dv d\omega) \right]^{\frac{\theta}{\theta-1}}$$

$$= \left[\int_{\Omega} \int_{\mathcal{V}(\omega)} \left[\left(\frac{P_t(v,\omega)}{P_t(v',\omega')} \right)^{-\theta} c_t(v',\omega') \right]^{\frac{\theta-1}{\theta}} \lambda(dv d\omega) \right]^{\frac{\theta}{\theta-1}}$$

$$= P_t(v',\omega')^{\theta} c_t(v',\omega') \left[\int_{\Omega} \int_{\mathcal{V}(\omega)} P_t(v,\omega)^{1-\theta} \lambda(dv d\omega) \right]^{\frac{\theta}{\theta-1}} .$$

First-Order Conditions VII

▸ Back to HH Problem ▸ Back to Firm Problem

Now recall $1 = P_t C_t$, and define $p_t(v,\omega) := P_t(v,\omega)/P_t$. The above expressions imply the following price index and demand curve:

$$1 = \left[\int_{\Omega} \int_{\mathcal{V}(\omega)} \left[p_t(v,\omega) \right]^{1-\theta} \lambda(dv d\omega) \right]^{\frac{1}{1-\theta}}, \tag{30}$$

$$c_t(v,\omega) = \left[p_t(v,\omega) \right]^{-\theta} C_t. \tag{31}$$

Optimality conditions for consumption goods producers. Consider firm ω's profit maximization problem (4). Eliminate constraints by using (1) and (31) to substitute for $p_t(v,\omega)$ and $y_t(v,\omega)$ in the firm-vintage profit function (2) that appears in (4). Obtain first-order optimality conditions by equating with zero the first derivatives of $\Pi_t(\omega)$ with respect to choice variables $k_t(v,\omega)$ and $l_t(v,\omega)$ for arbitrary vintage v. Firm ω's optimal choice of capital for production with vintage v satisfies

$$k_t(v,\omega) = (\alpha) \left(\frac{\theta - 1}{\theta} \right) (Y_t)^{\frac{1}{\theta}} \left[y_t(v,\omega) \right]^{\frac{\theta-1}{\theta}} (r_t)^{-1}. \tag{32}$$

First-Order Conditions VIII

Back to HH Problem Back to Firm Problem

Its optimal choice of labor satisfies

$$l_t(v,\omega) = (1-\alpha)\left(\frac{\theta-1}{\theta}\right)(Y_t)^{\frac{1}{\theta}}\left[y_t(v,\omega)\right]^{\frac{\theta-1}{\theta}}(w_t)^{-1}. \tag{33}$$

Notice that the optimal capital-labor ratio depends neither on the individual firm nor on the vintage of technology:

$$\frac{k_t(v,\omega)}{l_t(v,\omega)} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{w_t}{r_t}\right). \tag{34}$$

Optimality conditions for capital goods producers. Now consider the profit maximization problem for the capital goods producer. Take derivatives of gross profit with respect to the factors to obtain first-order conditions:

$$r_t = \alpha Z_t(k_t)^{\alpha-1}(l_t)^{1-\alpha}, \tag{35}$$

$$w_t = (1-\alpha)Z_t(k_t)^{\alpha}(l_t)^{-\alpha}. \tag{36}$$

Notice that the the capital-labor ratio in the capital goods sector is again

$$\frac{k_t}{l_t} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{w_t}{r_t}\right). \tag{37}$$

Derivation: Proposition 1 I

▸ Back to proposition

The household and capital goods producer are representative, so aggregation pertains only to the final goods sector.

Start with the optimality conditions (32) and (33) from the firm's decision problem (4). These expressions contain vintage-specific variables $k_t(v,\omega)$, $l_t(v,\omega)$, and $y_t(v,\omega)$ as well as variables and parameters common to all vintages. Combine equations (32) and (33) with the production function (1) to obtain expressions for $k_t(v,\omega)$, $l_t(v,\omega)$, and $y_t(v,\omega)$ in terms of $z_t(v)$ and variables and parameters common to all vintages:

$$k_t(v,\omega) = [z(\omega)z_t(v)]^{\theta-1}(Y_t)\left(\frac{\theta-1}{\theta}\right)^{\theta}\left(\frac{r_t}{\alpha}\right)^{\alpha(1-\theta)-1}\left(\frac{w_t}{1-\alpha}\right)^{(1-\alpha)(1-\theta)}, \tag{38}$$

$$l_t(v,\omega) = [z(\omega)z_t(v)]^{\theta-1}(Y_t)\left(\frac{\theta-1}{\theta}\right)^{\theta}\left(\frac{r_t}{\alpha}\right)^{\alpha(1-\theta)}\left(\frac{w_t}{1-\alpha}\right)^{(1-\alpha)(1-\theta)-1}, \tag{39}$$

$$y_t(v,\omega) = [z(\omega)z_t(v)]^{\theta}(Y_t)\left(\frac{\theta-1}{\theta}\right)^{\theta}\left(\frac{r_t}{\alpha}\right)^{-\alpha\theta}\left(\frac{w_t}{1-\alpha}\right)^{-(1-\alpha)\theta}. \tag{40}$$

Derivation: Proposition 1 II

Back to proposition

These expressions can be simplified further using an expression derived from the definition of the consumption basket, along with (40) and market clearing:

$$Y_t = \left[\int_{\Omega} \int_{\mathcal{V}(\omega)} [y_t(v,\omega)]^{\frac{\theta-1}{\theta}} \lambda(dv d\omega) \right]^{\frac{\theta}{\theta-1}}$$

$$= \left(\frac{\theta-1}{\theta}\right)^{\theta} \left(\frac{\alpha}{r_t}\right)^{\alpha\theta} \left(\frac{1-\alpha}{w_t}\right)^{(1-\alpha)\theta} (Y_t) \left[\int_{\Omega} \int_{\mathcal{V}(\omega)} (z(\omega) z_t(v))^{\theta-1} \lambda(dv d\omega) \right]^{\frac{\theta}{\theta-1}}$$

$$\Leftrightarrow \qquad Z_t := \left[\int_{\Omega} \int_{\mathcal{V}(\omega)} (z(\omega) z_t(v))^{\theta-1} \lambda(dv d\omega) \right]^{\frac{1}{\theta-1}} = \left(\frac{\theta}{\theta-1}\right) \left(\frac{r_t}{\alpha}\right)^{\alpha} \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}.$$

Derivation: Proposition 1 III

Back to proposition

Now use the expression for Z_t to simplify (38)–(40):

$$k_t(v,\omega) = \left(\frac{\theta-1}{\theta}\right)\left(\frac{\alpha}{r_t}\right)\left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta-1} Y_t$$
$$l_t(v,\omega) = \left(\frac{\theta-1}{\theta}\right)\left(\frac{1-\alpha}{w_t}\right)\left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta-1} Y_t$$
$$y_t(v,\omega) = \left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta} Y_t.$$

Now recall that $p_t(v,\omega) = (y_t(v,\omega)/Y_t)^{-(1/\theta)}$, and use above to get a similar expression for profit:

$$\begin{aligned}\pi_t(v,\omega) &= p_t(v,\omega)y_t(v,\omega) - r_t k_t(v,\omega) - w_t l_t(v,\omega) \\ &= \frac{1}{\theta}\left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta-1} Y_t.\end{aligned}$$

Derivation: Proposition 1 IV

Back to proposition

To get firm aggregates, sum the $k_t(v,\omega)$'s, $l_t(v,\omega)$'s, and $\pi_t(v,\omega)$'s, and use the Dixit-Stiglitz aggregator on $y_t(v,\omega)$:

$$K_t(\omega) := \int_{\mathcal{V}(\omega)} k_t(v,\omega)\lambda(dv) = \left(\frac{\theta-1}{\theta}\right)\left(\frac{\alpha}{r_t}\right)\left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta-1} Y_t$$

$$L_t(\omega) := \int_{\mathcal{V}(\omega)} l_t(v,\omega)\lambda(dv) = \left(\frac{\theta-1}{\theta}\right)\left(\frac{1-\alpha}{w_t}\right)\left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta-1} Y_t$$

$$Y_t(\omega) := \left[\int_{\mathcal{V}(\omega)} (y_t(v,\omega))^{\frac{\theta-1}{\theta}}\lambda(dv)\right]^{\frac{\theta}{\theta-1}} = \left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta} Y_t$$

$$\Pi_t(\omega) := \int_{\mathcal{V}(\omega)} \pi_t(v,\omega)\lambda(dv) = \frac{1}{\theta}\left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta-1} Y_t,$$

Derivation: Proposition 1 V

Back to proposition

where

$$Z_t(\omega) := \left[\int_{\mathcal{V}(\omega)} (z(\omega) z_t(v))^{\theta-1} \lambda(dv) \right]^{\frac{1}{\theta-1}}.$$

Further rearrangement along the same lines yields the economy-wide aggregates. It is also possible to write aggregate output in terms of a Cobb-Douglas aggregate production function, at both the firm and economy-wide levels:

$$Y_t(\omega) = Z_t(\omega)[K_t(\omega)]^{\alpha}[L_t(\omega)]^{1-\alpha} \quad (41)$$

$$Y_t = Z_t(K_t)^{\alpha}(L_t)^{1-\alpha}, \quad (42)$$

where the production function arguments should be understood as *optimal* factor inputs that satisfy the firm's optimality conditions for from the profit maximization problem (see Felipe & Fisher, 2003, for a discussion).

Derivation: Proposition 1 VI

Back to proposition

Notice that the firm-level aggregate production function takes the familiar Cobb-Douglas form. But remember that the distribution of shocks is endogenous, and the underlying technology choice problem imposes additional structure on the firm-level productivity multipliers. In particular, if technology sets $\mathcal{V}(\omega)$ differs across firms, so too will the distributions of the random productivity multipliers. And to the extent that technology sets share common elements, firm-level productivity will covary. The next three propositions make these statements rigorous.

Derivation: Proposition 2 I

▸ Back to proposition

Firms choose their technology sets $\mathcal{V}(\omega) \subseteq \mathcal{V} = [\underline{v}, \infty) \subseteq \mathbb{R}^+$ to maximize profit. Recall that technologies differ in their period fixed costs, but not their first two moments. Starting from the technology adoption rule in (5), and rearranging:

$$\begin{aligned}
0 &< \mathrm{E}_t \left[m_{t,t+1}(\pi_t(v,\omega) - f_s(v)) \right] \\
&= \mathrm{E}_t \left[\beta \frac{u'(C_{t+1})}{u'(C_t)} (\pi_t(v,\omega) - f_s(v)) \right] \\
&= \mathrm{E}_t \left[\frac{1}{Y_{t+1}} (\pi_{t+1}(v,\omega) - f_{t+1}(v)) \right],
\end{aligned}$$

where the third line assumes log utility. Now recall from the proof to 1:

$$\begin{aligned}
\pi_t(v,\omega) &= \frac{1}{\theta} \left(\frac{z(\omega) z_t(v)}{Z_t} \right)^{\theta - 1} Y_t, \\
f_t(v) &= \frac{Y_t}{\mu} v^{\gamma}.
\end{aligned}$$

Derivation: Proposition 2 II

▸ Back to proposition

Using these expressions in the adoption rule:

$$\mathrm{E}_t\left[\frac{1}{Y_{t+1}}(\pi_{t+1}(v,\omega)-f_{t+1}(v))\right]>0$$

$$\Leftrightarrow \qquad \left(\frac{z(\omega)^{\theta-1}}{\theta}\right)\mathrm{E}_t\left[\left(\frac{z_t(v)}{Z_t}\right)^{\theta-1}\right]\geq\frac{v^{\gamma}}{\mu}.$$

From here, either evaluate the productivities in the ratio under the expectation operator at their expected values to get an expression describing steady-state technology sets, or take an approximation of the expression under the expectation operator. A first-order approximation gives the same results as the steady-state solution:

$$\overline{v}(\omega)=\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\gamma}}z(\omega)^{\frac{\theta-1}{\gamma}}$$

$$\Rightarrow \qquad \underline{z}(v)=\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\theta-1}}v^{\frac{\gamma}{\theta-1}}.$$

Derivation: Proposition 2 III

▸ Back to proposition

Notice that the cut-off $\overline{v}(\omega)$ increasing in $z(\omega)$, so the more productive firms produce more varieties and use more technology.

Two remarks are in order: First, it is useful that the steady-state and first-order approximate cut-offs coincide, because it means that first-order dynamics around the steady state are completely standard in this model. Second, the second-order approximate case gives more interesting but less tractable results. There is a covariance term in the second-oder approximation that varies with v—covariance is higher for commonly-used technologies.

Derivation: Proposition 3 I

Back to proposition

Begin with the first moment of sector-aggregate productivity, just using the definition:

$$\mu = \mathrm{E}\left[Z_t^{\theta-1}\right] = \mathrm{E}\left[\int_{\Omega} Z_t(\omega)^{\theta-1}\lambda(d\omega)\right]$$

$$= \mathrm{E}\left[\int_{\Omega}\int_{\mathcal{V}(\omega)} (z(\omega)z_t(v))^{\theta-1}\lambda(dvd\omega)\right]$$

$$= \mathrm{E}\left[\int_{\mathcal{V}}\int_{\Omega_v} (z(\omega)z_t(v))^{\theta-1}\lambda(d\omega dv)\right] = \mathrm{E}\left[\int_{\mathcal{V}} z_t(v)^{\theta-1}\left(\int_{\Omega_v} z(\omega)^{\theta-1}\lambda(d\omega)\right)\lambda(d\omega)\right],$$

where Ω_v is the set of firms using vintage v, that is: $\Omega_v := \{\omega \in \Omega : \underline{z}(v) < z(\omega)\}$, and $\underline{z}(v)$ is the inverse of the cost cut-off $\overline{v}(\omega)$.

Derivation: Proposition 3 II

Back to proposition

Now evaluate the inner integral:

$$\int_{\Omega_v} z(\omega)^{\theta-1}\lambda(d\omega) = \int_{\underline{z}(v)}^{\infty} z(\omega)^{\theta-1} h(z(\omega))dz(\omega)$$
$$= \left[\frac{\kappa}{(\theta-1)-\kappa} z(\omega)^{(\theta-1)-\kappa}\right]_{\underline{z}(v)}^{\infty}$$
$$= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\underline{z}(v)^{(\theta-1)-\kappa}$$
$$= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}}\left(\frac{1}{v}\right)^{\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}.$$

Derivation: Proposition 3 III

▸ Back to proposition

Substitute the evaluated integral back into the expression for μ:

$$\mu = \mathrm{E}\left[Z_t^{\theta-1}\right] = \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}} \mathrm{E}\left[\int_{\mathcal{V}} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv)\right]$$

$$= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}} \mathrm{E}\left[\int_{\underline{v}}^{\infty} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv)\right]$$

Derivation: Proposition 3 IV

Back to proposition

Use the definition of technological productivity $z_t(v) := \epsilon_{t,\lceil v \rceil}$, set $\underline{v} = 1$, and write the remaining integral as:

$$
\begin{aligned}
\mathrm{E}\left[\int_{\underline{v}}^{\infty} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv)\right] &= \mathrm{E}\left[\int_{\underline{v}}^{\infty} \epsilon_{t,\lceil v \rceil}^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv)\right] \\
&= \mathrm{E}\left[\int_{1}^{2} \epsilon_{t,2}^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \int_{2}^{3} \epsilon_{t,3}^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \dots\right] \\
&= \mu_\epsilon \int_{1}^{2} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \mu_\epsilon \int_{2}^{3} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \dots.
\end{aligned}
$$

Derivation: Proposition 3 V

▸ Back to proposition

Now consider the integrals of the form:

$$\int_{n}^{n+1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) = \left[\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)} \right) v^{\frac{-\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} \right]_{n}^{n+1}$$

$$= \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)} \right) \left[\left(\frac{1}{n} \right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1} \right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} \right].$$

Derivation: Proposition 3 VI

▸ Back to proposition

Returning to the expression for μ:

$$\mu = \mathrm{E}\left[Z_t^{\theta-1}\right] = \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}} \mu_\epsilon \sum_{n=1}^{\infty}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)$$
$$\times\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right]$$
$$= \mu_\epsilon\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}}.$$

Notice that $\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\theta-1}}$ appears on the right-hand side. Substituting it for $\underline{z}$, and collecting parameters,

$$\mu = \mu_\epsilon \zeta_{\mu 1} \underline{z}^{\zeta_{\mu 2}},$$

Derivation: Proposition 3 VII

Back to proposition

where

$$\zeta_{\mu 1} := \left(\frac{\theta - 1}{\gamma[\kappa - (\theta - 1)] - (\theta - 1)}\right)\left(\frac{\kappa}{\kappa - (\theta - 1)}\right)$$
$$\zeta_{\mu 2} := \kappa - (\theta - 1)$$

Derivation: Proposition 3 VIII

Back to proposition

Now turn to the second moment of sector-aggregate productivity. Starting again with the definition:

$$
\begin{aligned}
\sigma^2 = \mathrm{Var}\left(Z_t^{\theta-1}\right) &= \mathrm{Var}\left(\int_{\Omega} Z_t(\omega)^{\theta-1}\lambda(d\omega)\right) = \mathrm{Var}\left(\int_{\Omega}\int_{\mathcal{V}(\omega)} (z(\omega)z_t(v))^{\theta-1}\lambda(dv d\omega)\right) \\
&= \mathrm{Var}\left(\int_{\mathcal{V}} z_t(v)^{\theta-1}\int_{\Omega_v} z(\omega)^{\theta-1}\lambda(d\omega dv)\right) \\
&= \mathrm{Var}\left(\int_{\mathcal{V}} z_t(v)^{\theta-1}\int_{\underline{z}(v)}^{\infty} z(\omega)^{\theta-1}\frac{\kappa}{z(\omega)^{\kappa+1}}\lambda(dz(\omega)dv)\right) \\
&= \mathrm{Var}\left(\int_{\mathcal{V}} z_t(v)^{\theta-1}\frac{\kappa}{(\theta-1)-\kappa}\underline{z}(v)^{-[\kappa-(\theta-1)]}\lambda(d\omega)\right),
\end{aligned}
$$

Derivation: Proposition 3 IX

Back to proposition

where from the third to the fourth line I change measure from Lebesgue to Pareto. Continuing, using $\underline{z}(v) = \left(\frac{\theta}{\mu_\epsilon}\right)^{\frac{1}{\theta-1}} v^{\frac{\gamma}{\theta-1}}$,

$$\sigma^2 = \mathrm{Var}\left(Z_t^{\theta-1}\right) = \mathrm{Var}\left(\int_{\mathcal{V}} \left(\frac{\kappa}{(\theta-1)-\kappa}\right)\left(\frac{\theta}{\mu_\epsilon}\right)^{-\frac{\kappa-(\theta-1)}{\theta-1}} z_t(v)^{\theta-1} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(d\omega)\right)$$

$$= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)^2 \left(\frac{\theta}{\mu_\epsilon}\right)^{-2\frac{\kappa-(\theta-1)}{\theta-1}} \mathrm{Var}\left(\int_{\mathcal{V}} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}\right).$$

Derivation: Proposition 3 X

Back to proposition

Now consider the integral:

$$\int_{\mathcal{V}} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(d\omega) = \int_{\mathcal{V}} \epsilon_{t,\lceil v \rceil} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(d\omega) = \int_{\overline{v}(\omega)}^{\infty} \epsilon_{t,\lceil v \rceil} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(d\omega)$$

$$= \epsilon_{t,2} \int_{1}^{2} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(d\omega) + \epsilon_{t,3} \int_{2}^{3} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(d\omega) + \dots$$

$$= \sum_{n=1}^{\infty} \frac{\epsilon_{t,n+1}(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)} \times \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} \right].$$

Derivation: Proposition 3 XI

▸ Back to proposition

Returning to the expression for σ^2:

$$
\begin{aligned}
\sigma^2 = \mathrm{Var}\left(Z_t^{\theta-1}\right) &= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)^2 \left(\frac{\theta}{\mu_\epsilon}\right)^{-2\,\frac{\kappa-(\theta-1)}{\theta-1}} \\
&\quad \times \mathrm{Var}\left(\sum_{n=1}^{\infty} \frac{\epsilon_{t,n+1}(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)} \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} \quad \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}}\right]\right) \\
&= \sigma_\epsilon^2 \left(\frac{\kappa}{\kappa-(\theta-1)}\right)^2 \left(\frac{\theta}{\mu_\epsilon}\right)^{-2\,\frac{\kappa-(\theta-1)}{\theta-1}} \left(\frac{(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)^2 \\
&\quad \times \sum_{n=1}^{\infty} \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}}\right]^2 .
\end{aligned}
$$

Derivation: Proposition 3 XII

▸ Back to proposition

Notice that $\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\theta-1}}$ appears on the right-hand side. Substituting it for $\underline{z}$, and collecting parameters,

$$\sigma^2 = \sigma_\epsilon^2 \zeta_{\sigma 1} \underline{z}^{\zeta_{\sigma 2}},$$

where

$$\zeta_{\sigma 1} := \left(\frac{\kappa}{\kappa - (\theta - 1)}\right)^2 \left(\frac{(\theta - 1)}{\gamma[\kappa - (\theta - 1)] - (\theta - 1)}\right)^2 \sum_{n=1}^{\infty} \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa - (\theta - 1)] - (\theta - 1)}{\theta - 1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa - (\theta - 1)] - (\theta - 1)}{\theta - 1}}\right]^2$$

$$\zeta_{\sigma 2} := 2\left[\kappa - (\theta - 1)\right].$$

Intuition for finite, non-zero variance: It's all about the weights

Back to proposition

Let $X := \sum_{n=1}^{N} a_n \epsilon_n$, with iid ϵ_n's and $\sigma_\epsilon^2 < \infty$.

$$\text{Var}(X) = \text{Var}\left(\sum_{n=1}^{N} a_n \epsilon_n\right) = \sum_{n=1}^{N} a_n^2 \sigma_\epsilon^2$$

Examples:

- $a_n = \frac{1}{N} \quad \Rightarrow \quad \text{Var}(X) = \frac{\sigma_\epsilon^2}{N}$ (Gabaix (2011)'s simplest example . . .)
- $a_n = \left(\frac{1}{N}\right)^2 \quad \Rightarrow \quad \text{Var}(X) = \frac{\sigma_\epsilon^2}{N^3}$ (Faster convergence to zero . . .)
- $a_n = \left(\frac{1}{N}\right)^{\frac{1}{2}} \quad \Rightarrow \quad \text{Var}(X) = \sigma_\epsilon^2$ (Doesn't converge to zero . . .)
- $a_n = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{o.w.} \end{cases} \quad \Rightarrow \quad \text{Var}(X) = \sigma_\epsilon^2$ (Weight bunched on one element . . .)

So what are weights in my set-up? All bunched up. . .

▸ Back to proposition

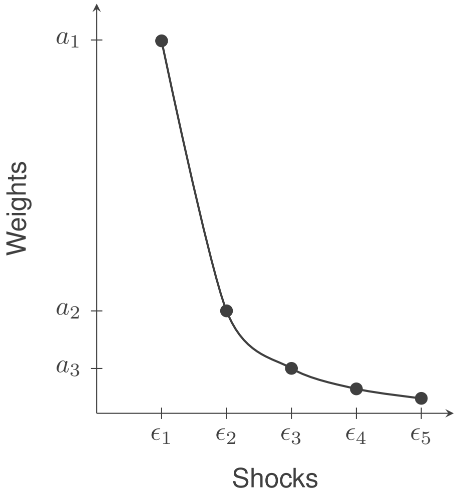
What determines weights here?

- Density and productivity of firms
- Set of firms using v shrinks fast as $v \uparrow$
- Faster if γ or κ large

Weights on technology shocks are of form:

$$a_n = \left(\frac{1}{\zeta}\right)^2 \left[\left(\frac{1}{n}\right)^\zeta - \left(\frac{1}{n+1}\right)^\zeta\right]^2$$

where $\zeta = 5$ — e.g., $\gamma = 6$, $\theta = 3.5$, $\kappa = 4$.



Back

Derivation: Proposition 4 I

To start, identify a specific firm ω_1, use the definitions of $Z_t(\omega_1)$ and Z_t in the covariance expression, and the cut-offs $\underline{z}$ and $\overline{v}(\omega)$ for the integral bounds:

$$\begin{aligned}\sigma_{\omega\Omega}(\omega) = \mathrm{Cov}\left(Z_t(\omega_1)^{\theta-1}, Z_t^{\theta-1}\right) &= \mathrm{Cov}\left(\int\limits_{\mathcal{V}_t(\omega_1)} [z(\omega_1)z_t(v)]^{\theta-1}\lambda(dv), \int\limits_{\Omega} Z_t(\omega)^{\theta-1}\lambda(d\omega)\right) \\ &= \mathrm{Cov}\left(\int\limits_{\underline{v}=1}^{\overline{v}(\omega_1)} [z(\omega)z_t(v)]^{\theta-1}\lambda(dv), \int\limits_{\underline{z}}^{\infty}\int\limits_{\underline{v}=1}^{\overline{v}(\omega)} [z(\omega)z_t(v)]^{\theta-1}\lambda(dvd\omega)\right).\end{aligned}$$

▸ Back

Derivation: Proposition 4 II

Now consider the first integral:

$$
\begin{aligned}
\int_{\underline{v}=1}^{\overline{v}(\omega_1)} [z(\omega_1) z_t(v)]^{\theta-1} \lambda(dv) &= z(\omega_1)^{\theta-1} \int_{\underline{v}=1}^{\overline{v}(\omega_1)} \epsilon_{t,\lceil v \rceil} \lambda(dv) \\
&= z(\omega_1)^{\theta-1} \left[\int_1^2 \epsilon_{t,2} \lambda(dv) + \int_2^3 \epsilon_{t,3} \lambda(dv) + \cdots + \int_{\overline{v}(\omega)-1}^{\overline{v}(\omega)} \epsilon_{t,\overline{v}(\omega)} \lambda(dv) \right] \\
&= z(\omega_1)^{\theta-1} \sum_{n=1}^{\overline{v}(\omega)-1} \epsilon_{t,n+1},
\end{aligned}
$$

where I have assumed w.l.g. that $\overline{v}(\omega) \in \mathbb{N}$.

Derivation: Proposition 4 III

Now consider the second integral:

$$\int_{\underline{z}}^{\infty}\int_{\underline{v}=1}^{\overline{v}(\omega)} [z(\omega)z_t(v)]^{\theta-1}\lambda(dvd\omega) = \int_{\underline{v}=1}^{\infty} z_t(v)^{\theta-1}\left(\int_{\underline{z}(v)}^{\infty} z(\omega)^{\theta-1}\lambda(d\omega)\right)\lambda(dv)$$

$$= \int_{\underline{v}}^{\infty} z_t(v)^{\theta-1}\left(\int_{\underline{z}(v)}^{\infty} z(\omega)^{\theta-1}h(z(\omega))\lambda(dz(\omega))\right)\lambda(dv)$$

$$= \int_{\underline{v}}^{\infty} z_t(v)^{\theta-1}\frac{\kappa}{\kappa-(\theta-1)}\underline{z}(v)^{-[\kappa-(\theta-1)]}\lambda(dv),$$

Back

Derivation: Proposition 4 IV

where line two changes measure from Lebesgue to Pareto. Continuing with the second integral, using $\underline{z}(v) = \left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{1}{\theta-1}} v^{\frac{\gamma}{\theta-1}}$,

$$\int_{\underline{z}}^{\infty} \int_{\underline{v}=1}^{\overline{v}(\omega)} [z(\omega) z_t(v)]^{\theta-1} \lambda(dv d\omega) = \left(\frac{\kappa}{\kappa-(\theta-1)}\right) \left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}} \int_{\underline{v}}^{\infty} z_t(v)^{\theta-1} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv).$$

Now the single integral on the right-hand side:

$$\begin{aligned} \int_{\underline{v}}^{\infty} z_t(v)^{\theta-1} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) &= \int_{\underline{v}=1}^{\infty} \epsilon_{t,\lceil v \rceil}^{\theta-1} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) \\ &= \epsilon_{t,2} \int_{1}^{2} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \epsilon_{t,3} \int_{2}^{3} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \dots \end{aligned}$$

.

Back

Derivation: Proposition 4 V

Now consider the integrals of the form:

$$\int_{n}^{n+1} v^{\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) = \left[\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) v^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right]_{n}^{n+1}$$

$$= \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right)\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right].$$

So the single integral becomes:

$$\int_{\underline{v}}^{\infty} z_t(v)^{\theta-1} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) = \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right)$$

$$\times \sum_{n=1}^{\infty} \epsilon_{t,n+1}\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right],$$

Back

Derivation: Proposition 4 VI

and the second integral becomes:

$$\int_{\underline{z}}^{\infty}\int_{\underline{v}=1}^{\overline{v}(\omega)}[z(\omega)z_t(v)]^{\theta-1}\lambda(dvd\omega)=\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right)$$

$$\times\sum_{n=1}^{\infty}\epsilon_{t,n+1}\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}-\left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right].$$

Back

Derivation: Proposition 4 VII

Now, recall that $\text{Cov}\left(\epsilon_{t,n},\epsilon_{t,m}\right)=0\;\forall\; n\neq m$, and write the desired covariance as:

$$\begin{aligned}
\sigma_{\omega\Omega}(\omega) &= \text{Cov}\left(Z_t(\omega_1)^{\theta-1}, Z_t^{\theta-1}\right)\\
&= z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right)\\
&\times \text{Cov}\left(\sum_{n=1}^{\overline{v}(\omega)-1}\epsilon_{t,n+1}, \sum_{n=1}^{\infty}\epsilon_{t,n+1}\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}-\left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right]\right)\\
&= z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right)\\
&\quad\times\sum_{n=1}^{\overline{v}(\omega)-1}\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}-\left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right]\text{Cov}\left(\epsilon_{t,n+1},\epsilon_{t,n+1}\right)
\end{aligned}$$

,

Derivation: Proposition 4 VIII

where $\text{Cov}\left(\epsilon_{t,n+1}, \epsilon_{t,n+1}\right) = \sigma_\epsilon^2$.
Notice that the right-hand side summation, with a as a temporary placeholder, is of form:

$$\sum_{n=1}^{\overline{v}(\omega_1)-1}\left[\left(\frac{1}{n}\right)^a - \left(\frac{1}{n+1}\right)^a\right] = \left[\left(\frac{1}{1}\right)^a - \left(\frac{1}{2}\right)^a + \left(\frac{1}{2}\right)^a - \left(\frac{1}{3}\right)^a + \cdots - \left(\frac{1}{\overline{v}(\omega_1)}\right)^a\right]$$
$$= \left[1 - \left(\frac{1}{\overline{v}(\omega_1)}\right)^a\right].$$

Back

Derivation: Proposition 4 IX

Returning to the covariance expression, and simplifying the summation as above,

$$\begin{aligned}
\sigma_{\omega\Omega}(\omega) &= \mathrm{Cov}\left(Z_t(\omega_1)^{\theta-1}, Z_t^{\theta-1}\right) \\
&= \sigma_\epsilon^2 z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\
&\quad\times\left[1-\left(\frac{1}{\overline{v}(\omega_1)}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right] \\
&= \sigma_\epsilon^2 z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\
&\quad\times\left[1-\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{\gamma[\kappa-(\theta-1)]}{\gamma(\theta-1)}}\left(\frac{1}{z(\omega_1)}\right)^{\frac{\gamma[\kappa-(\theta-1)]}{\gamma}}\right]
\end{aligned}$$

where the last line uses $\overline{v}(\omega_1) = \left(\frac{\mu_\epsilon}{\theta\mu}\right)^{\frac{1}{\gamma}} z(\omega_1)^{\frac{\theta-1}{\gamma}}$.

▸ Back Derivation: Proposition 4 X

Finally, collect parameters, return from specific ω_1 to arbitrary ω, and write:

$$\frac{\sigma_{\omega\Omega}(\omega)}{\sigma_\epsilon^2} = z(\omega)^{\theta-1}\zeta_{\omega\Omega1}\left[1-\left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega2}}\right]\text{, where}$$

$$\zeta_{\omega\Omega1} = \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1)]}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right)$$

$$\zeta_{\omega\Omega1} = \frac{\gamma[\kappa-(\theta-1)]}{\gamma}.$$

Recall that the μ appearing in $\zeta_{\omega\Omega1}$ has already been expressed in terms of parameters, so the above expression suffices.

Back Derivation: Proposition 4 XI

Now turn to covariance over market value. Start from the following primitives:

$$V_t(\omega) = \mathrm{E}_t\left[\sum_{s=t+1}^{\infty} \beta^{s-t} \frac{u'(C_s)}{u'(C_t)} \big(\Pi_s(\omega) - F_s(\omega)\big)\right]$$

$$\Pi_s(\omega) = \int_{\mathcal{V}(\omega)} \pi_s(v,\omega)\lambda(dv) = \frac{1}{\theta}\left(\frac{Z_s(\omega)}{Z_s}\right)^{\theta-1} Y_s$$

$$F_s(\omega) = \int_{\mathcal{V}(\omega)} \frac{Y_s}{\mu} v^{\gamma}\lambda(dv).$$

Using $u(C_s) = \ln(C_s)$ and above primitives, rearrange to get:

$$\frac{V_t(\omega)}{Y_t} = \mathrm{E}\left[\sum_{s=t+1}^{\infty} \beta^{s-t}\left\{\frac{1}{\theta}\left(\frac{z(\omega)}{Z_s}\right)^{\theta-1} \int_{\mathcal{V}(\omega)} z_s(v)^{\theta-1} - \frac{v^{\gamma}}{\mu}\lambda(dv)\right\}\right].$$

▸ Back Derivation: Proposition 4 XII

Split up the integral and evaluate the first term, assuming w.l.g. that $\overline{v}(\omega) \in \mathbb{N}$:

$$\int_{\mathcal{V}(\omega)} z_s(v)^{\theta-1} \lambda(dv) = \int_{\underline{v}=1}^{\overline{v}(\omega)} \epsilon_{s,\lceil v \rceil} \lambda(dv)$$

$$= \int_1^2 \epsilon_{s,2} \lambda(dv) + \int_2^3 \epsilon_{s,3} \lambda(dv) + \cdots + \int_{\overline{v}(\omega)-1}^{\overline{v}(\omega)} \epsilon_{s,\overline{v}(\omega)} \lambda(dv)$$

$$= \sum_{n=1}^{\overline{v}(\omega)-1} \epsilon_{s,n+1}$$

Now evaluate the second part of the integral that we split above:

$$\int_{\mathcal{V}(\omega)} \frac{v^\gamma}{\mu} \lambda(dv) = \frac{1}{\mu} \left(\frac{\overline{v}(\omega)^{\gamma+1}}{\gamma+1} - \frac{1}{\gamma+1} \right).$$

Back

Derivation: Proposition 4 XIII

Substituting back into the expression for firm value,

$$\frac{V_t(\omega)}{Y_t} = \sum_{s=t+1}^{\infty} \beta^{s-t} \frac{z(\omega)^{\theta-1}}{\theta} \sum_{n=1}^{\overline{v}(\omega)-1} \mathrm{E}\left[\frac{\epsilon_{s,n+1}}{Z_s^{\theta-1}}\right] - \sum_{s=t+1}^{\infty} \beta^{s-t} \left(\frac{\overline{v}(\omega)^{\gamma+1}}{1+\gamma} - \frac{1}{1+\gamma}\right)$$

To a first-order approximation, the expectation is: $\mathrm{E}\left[\frac{\epsilon_{s,n+1}}{Z_s^{\theta-1}}\right] \approx \frac{\mu_\epsilon}{\mu}$. Simplifying,

$$\frac{V_t(\omega)}{Y_t} \approx z(\omega)^{\frac{1+\gamma}{\gamma}(\theta-1)} \left(\frac{\mu_\epsilon}{\theta\mu}\right)^{\frac{1+\gamma}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right) - z(\omega)^{\theta-1}\left(\frac{\mu_\epsilon}{\theta\mu}\right) + \left(\frac{1}{1+\gamma}\right)$$

Now combining with the covariance expression derived above:

$$\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)} \approx \frac{\sigma_\epsilon^2}{Y_t} \cdot \frac{z(\omega)^{\theta-1}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)\left[1-\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\gamma(\theta-1)}}\left(\frac{1}{z(\omega)}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\gamma}}\right]}{z(\omega)^{\frac{1+\gamma}{\gamma}(\theta-1)}\left(\frac{\mu_\epsilon}{\theta\mu}\right)^{\frac{1+\gamma}{\gamma}}\left(\frac{\gamma}{1+\gamma}\right) - z(\omega)^{\theta-1}\left(\frac{\mu_\epsilon}{\theta\mu}\right) + \frac{1}{1+\gamma}}.$$

Back Derivation: Proposition 4 XIV

Finally, using the expression for $\underline{z}$, and collecting parameters to simplify,

$$\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)} \approx \frac{1}{Y_t}\left(\frac{\zeta_{\omega\Omega 1}\left[1-\left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega 2}}\right]}{\zeta_{V1}\left(\frac{z(\omega)}{\underline{z}}\right)^{\zeta_{V2}}+\zeta_{V3}\left(\frac{1}{z(\omega)}\right)^{\zeta_{V4}}-\left(\frac{1}{\underline{z}}\right)^{\zeta_{V4}}}\right)\text{, where}$$

$$\zeta_{\omega\Omega 1} := \underline{z}^{\theta-1}\left(\frac{\sigma_\epsilon^2(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right), \qquad \zeta_{\omega\Omega 2} := \left(\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\gamma}\right)$$

$$\zeta_{V1} = \left(\frac{\gamma}{1+\gamma}\right), \qquad \zeta_{V2} := \left(\frac{\theta-1}{\gamma}\right), \qquad \zeta_{V3} := \left(\frac{1}{\gamma+1}\right), \qquad \zeta_{V4} := (\theta-1)\,.$$

Back

Derivation: Proposition 5 I

Start with the definition of firm ω's stock return:

$$
\begin{aligned}
r_{t+1}(\omega) &= \frac{V_{t+1}(\omega) + \Pi_{t+1}(\omega) - F_{t+1}(\omega)}{V_t(\omega)} \\
&= \frac{\mathrm{E}\left[\sum_{s=t+2}^{\infty} m_{t+1,s}(\Pi_s(\omega) - F_s(\omega))\right] + \Pi_{t+1}(\omega) - F_{t+1}(\omega)}{V_t(\omega)} \\
&= \frac{Y_{t+1}\mathrm{E}\left[\sum_{s=t+1}^{\infty} \beta^{s-(t+1)} \left(\frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s}\right)\right]}{V_t(\omega)},
\end{aligned}
$$

Derivation: Proposition 5 II

where the third line assumes log utility and uses the definition of the household stochastic discount factor. Now take the time-t conditional expectation:

$$\begin{aligned}
\mathrm{E}_t\left[r_{t+1}(\omega)\right] &= \mathrm{E}_t\left[\frac{Y_{t+1}\mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]}{V_t(\omega)}\right] \\
&= \frac{\mathrm{E}_t\left[Y_{t+1}\right]\mathrm{E}_t\left[\mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right]}{V_t(\omega)} \\
&\quad + \frac{\mathrm{Cov}_t\left(Y_{t+1},\mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right)}{V_t(\omega)} \\
&= \frac{\mathrm{E}_t\left[Y_{t+1}\right]}{\beta Y_t} + \frac{\mathrm{Cov}_t\left(Y_{t+1},\mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right)}{V_t(\omega)} \\
\Leftrightarrow \quad \mathrm{E}_t\left[r_{t+1}(\omega)-r_{f,t+1}\right] &= \frac{\mathrm{Cov}_t\left(Y_{t+1},\mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right)}{V_t(\omega)}.
\end{aligned}$$

▸ Back

Derivation: Proposition 5 III

Consider the covariance term separately, recalling that zero serial correlation is assumed for $\epsilon_{s,n}$s:

$$\mathrm{Cov}_t\left(Y_{t+1}, \mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty} \beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s}\right)\right]\right) = \mathrm{Cov}_t\left(Y_{t+1}, \left(\frac{\Pi_{t+1}(\omega)}{Y_{t+1}} - \frac{F_{t+1}(\omega)}{Y_{t+1}}\right)\right)$$

$$= \mathrm{E}_t\left[\Pi_{t+1}(\omega) - F_{t+1}(\omega)\right] - \mathrm{E}_t\left[Y_{t+1}\right]\mathrm{E}_t\left[\frac{\Pi_{t+1}(\omega)}{Y_{t+1}} - \frac{F_{t+1}(\omega)}{Y_{t+1}}\right]$$

$$= \mathrm{E}_t\left[Y_{t+1}\int_{\mathcal{V}(\omega)} \frac{1}{\theta}\left(\frac{z(\omega)z_{t+1}(v)}{Z_{t+1}}\right)^{\theta-1}\lambda(dv)\right] - \mathrm{E}_t\left[Y_{t+1}\right]\mathrm{E}_t\left[\int_{\mathcal{V}(\omega)} \frac{1}{\theta}\left(\frac{z(\omega)z_{t+1}(v)}{Z_{t+1}}\right)^{\theta-1}\lambda(dv)\right]$$

$$= \mathrm{E}_t\left[\frac{1}{\theta}Y_{t+1}\left(\frac{Z_{t+1}(\omega)}{Z_{t+1}}\right)^{\theta-1}\right] - \mathrm{E}_t\left[Y_{t+1}\right]\mathrm{E}_t\left[\frac{1}{\theta}\left(\frac{Z_{t+1}(\omega)}{Z_{t+1}}\right)^{\theta-1}\right],$$

Back

Derivation: Proposition 5 IV

where the second line uses the assumption of zero serial correlation. Now recall from (41) that $Y_{t+1} = Z_{t+1}(K_{t+1})^{\alpha}(L)^{1-\alpha}$, that K_{t+1} is determined in t, and that L fixed, so

$$\mathrm{Cov}_t\left(Y_{t+1}, \mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right)$$
$$=\frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left(\mathrm{E}_t\left[\frac{Z_{t+1}(\omega)^{\theta-1}}{Z_{t+1}^{\theta-2}}\right]-\mathrm{E}_t\left[Z_{t+1}\right]\mathrm{E}_t\left[\left(\frac{Z_{t+1}(\omega)}{Z_{t+1}}\right)^{\theta-1}\right]\right)$$

Next, second-order approximate the individual right-hand side expectations around the non-stochastic steady state values $\mu(\omega)$ and μ. Starting with the first right-hand side expectation:

$$\mathrm{E}_t\left[\frac{Z_{t+1}(\omega)^{\theta-1}}{Z_{t+1}^{\theta-2}}\right]\approx\frac{\mu(\omega)}{\mu^{\frac{\theta-2}{\theta-1}}}+\frac{1}{2}\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{\theta-2}{\theta-1}+1\right)\frac{\mu(\omega)}{\mu^{\frac{\theta-2}{\theta-1}+2}}\cdot\sigma^2-\frac{1}{2}\left(\frac{\theta-2}{\theta-1}\right)\frac{1}{\mu^{\frac{\theta-2}{\theta-1}+1}}\cdot\sigma_{\omega\Omega}(\omega).$$

Derivation: Proposition 5 V

Now the second:

$$\mathrm{E}_t\left[Z_{t+1}\right] \approx \mu^{\frac{1}{\theta-1}} + \frac{1}{2}\left(\frac{1}{\theta-1}\right)\left(\frac{1}{\theta-1}-1\right)\mu^{\frac{1}{\theta-1}-2}\sigma^2.$$

And the third:

$$\mathrm{E}_t\left[\left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta-1}\right] \approx \left(\frac{1}{\mu}+\frac{\sigma^2}{\mu^3}\right)\mu(\omega) - \left(\frac{1}{2\mu^2}\right)\sigma_{\omega\Omega}(\omega).$$

Back

Derivation: Proposition 5 VI

Substituting the approximations in the covariance expression, and rearranging,

$$
\begin{aligned}
&\mathrm{Cov}_t\left(Y_{t+1}, \mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right)\\
&\approx \frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left(\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{\mu^{\frac{1}{\theta-1}}}{\mu}\right)\sigma^2+\left(\frac{\mu^{\frac{1}{\theta-1}}}{\mu}\right)\left(\frac{\sigma}{\mu}\right)^2\left[\left(\frac{1}{2}\right)\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{1}{\theta-1}\right)\left(\frac{\sigma}{\mu}\right)^2-1\right]\right)\cdot\mu(\omega)\\
&\qquad+\frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left[\left(\frac{1}{2}\right)\left(\frac{1}{\theta-1}\right)\left(\frac{1}{\mu^{\frac{1}{\theta-1}}}\right)\left\{1-\left(\frac{1}{2}\right)\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{1}{\mu^2}\right)\right\}\right]\cdot\sigma_{\omega\Omega}(\omega)\\
&\approx \zeta_{r1}\mu(\omega)+\zeta_{r2}\sigma_{\omega\Omega}(\omega),
\end{aligned}
$$

▸ Back Derivation: Proposition 5 VII

where ζ_{r1} and ζ_{r2} are parameter collections given by:

$$\zeta_{r1} := \frac{(K_{t+1}^{\alpha} L^{1-\alpha})}{\theta} \left(\left(\frac{\theta-2}{\theta-1}\right) \left(\frac{\mu^{\frac{1}{\theta-1}}}{\mu}\right) \sigma^2 + \left(\frac{\mu^{\frac{1}{\theta-1}}}{\mu}\right) \left(\frac{\sigma}{\mu}\right)^2 \left[\left(\frac{1}{2}\right) \left(\frac{\theta-2}{\theta-1}\right) \left(\frac{1}{\theta-1}\right) \left(\frac{\sigma}{\mu}\right)^2 - 1 \right] \right)$$

$$\zeta_{r2} := \frac{(K_{t+1}^{\alpha} L^{1-\alpha})}{\theta} \left[\left(\frac{1}{2}\right) \left(\frac{1}{\theta-1}\right) \left(\frac{1}{\mu^{\frac{1}{\theta-1}}}\right) \left\{ 1 - \left(\frac{1}{2}\right) \left(\frac{\theta-2}{\theta-1}\right) \left(\frac{1}{\mu^2}\right) \right\} \right],$$

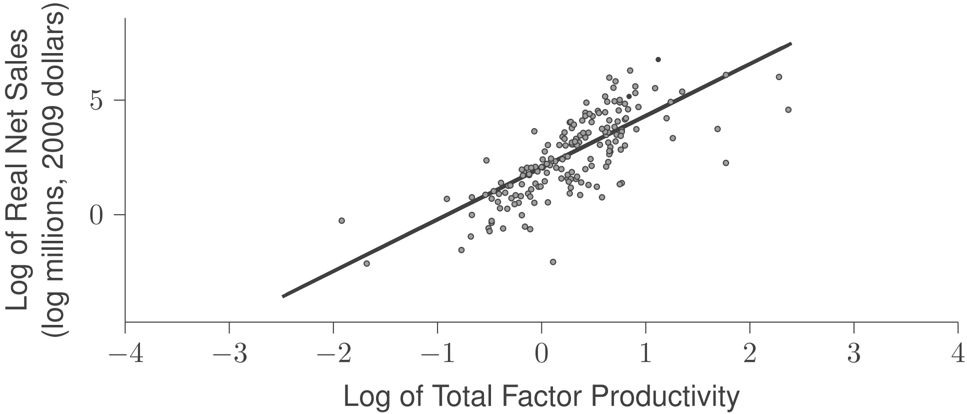
and K_{t+1} is evaluated at its steady-state value. Finally, returning to the expression for expected excess returns,

$$\mathrm{E}_t \left[r_{t+1}(\omega) - r_{f,t+1} \right] \approx \zeta_{r1} \frac{\mu(\omega)}{V_t(\omega)} + \zeta_{r2} \frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)}.$$

Appendix

Total factor productivity estimated using Olley and Pakes (1996)

Compustat Annual Fundamentals North America, 2015; pricolor dot are Starbucks and Boeing (left to right)



Traditional productivity estimates are problematic

Slides follow Olley and Pakes (1996) and Arnold (2005)

Traditional productivity estimation: $y_{i,t} = \beta_l l_{i,t} + \beta_k k_{i,t} + u_{i,t}$,

Problems:

1. Simultaneity bias
2. Selection bias

Notation:

$y_{i,t}$	log of value added at firm i
$l_{i,t}$	log of labor usage at firm i
$k_{i,t}$	log of capital usage at firm i
$u_{i,t}$	error term

Back to evidence

Traditional productivity estimates are problematic: simultaneity puts an upward bias on estimates I

Traditional productivity estimation: $y_{i,t} = \beta_l l_{i,t} + \beta_k k_{i,t} + u_{i,t}$,

Simultaneity: Explanatory and dependent variables influenced by $u_{i,t}$

- Firm i probably knows its productivity when choosing inputs (serial corr)
- If productivity rises, both output *and* inputs rise in response
- Left uncorrected, model attributes too much output variation to inputs
- Thus, upward bias in capital and labor coefficient estimates

Back to evidence

Traditional productivity estimates are problematic: selection puts a downward bias on estimates I

Traditional productivity estimation: $y_{i,t} = \beta_l l_{i,t} + \beta_k k_{i,t} + u_{i,t}$,

Selection: Sample not representative of population

- Firm entry and exit are common, firm panel data usually unbalanced
- Balancing the panel introduces bias because of exit selection
- Firms w/ low capital stock tend to exit after bad productivity shock
- Firms w/ high capital stock tend to survive, so corr(shock, stock) < 0
- Thus, downward bias in capital coefficient estimates

Back to evidence

Olley and Pakes (1996) estimate three equations that together take explicit account of simultaneity and selection bias I

Olley-Pakes productivity estimation:

$$P_{i,t} = \mathscr{P}_t(i_{i,t}, k_{i,t}) \tag{43}$$

$$y_{i,t} = \beta_l l_{i,t} + \phi(i_{i,t}, k_{i,t}) + \eta_t \tag{44}$$

$$y_{i,t+1} - \beta_l l_{i,t+1} = \beta_k k_{i,t+1} + g(P_{i,t}, \phi_{i,t} - \beta_k k_{i,t}) + \xi_{i,t+1} + \eta_{i,t+1} \tag{45}$$

where ϕ and $g(\cdot)$ estimated non-parametrically, used to control for biases

Olley and Pakes (1996) procedure takes explicit account of simultaneity and selection bias I

Outline of Procedure:

1. Split error term u into two components:

$$y_{i,t} = \beta_l l_{i,t} + \beta_k k_{i,t} + \omega_{i,t} + \eta_{i,t}, \tag{46}$$

   where ω, η unobserved, but firm can forecast ω

2. Use behavioral model to derive relationships: $\omega(i,k)$, $\underline{\omega}(k)$, $P(i,k)$
   where i is investment, P survival probability, $\underline{\omega}$ exit cut-off productivity

3. Use behavioral relations to transform (46) into semi-parametric form in (44)

4. Probit estimation of P in (43), approximating $\mathscr{P}$ non-parametrically

Olley and Pakes (1996) procedure takes explicit account of simultaneity and selection bias II

5. First stage: Estimation of β_l, ϕ in (44), approximating ϕ non-parametrically

6. Second stage: Use estimates of β_l, ϕ, P to estimate (45)

7. Fit (46), get unbiased productivity estimate from residual

Olley and Pakes (1996): Details on controlling for simultaneity I

Behavioral model implies (unspecified) function for optimal investment:

$$\begin{aligned} i_{i,t} &= i_t(\omega_i, t, k_{i,t}) \\ \Rightarrow \quad \omega_{i,t} &= h_t(i_{i,t}, k_{i,t}) \qquad \text{iff } i(\cdot) \text{ invertible.} \end{aligned}$$

Notice that $h(\cdot)$ relates unobservable ω to observables k and i.

Define $\phi_t(i_{i,t}, k_{i,t}) := \beta_k k_{i,t} + h_t(i_{i,t}, k_{i,t})$, approximate ϕ_t non-parametrically, transform (46) and estimate

$$y_{i,t} = \beta_l l_{i,t} + \phi_t(i_{i,t}, k_{i,t}) + \eta_{i,t}.$$

Olley and Pakes (1996): Details on controlling for selection (1/2) I

Behavioral model implies (unspecified) exit rule: exit if $\chi = 0$, where

$$\chi_{i,t} = \begin{cases} 1 & \text{if } \omega_{i,t} \geq \underline{\omega}_{i,t}(k_t), \\ 0 & \text{otherwise.} \end{cases}$$

Use probit to estimate probability of survival P, where:

$$\begin{aligned} \text{Pr}(\chi_t = 1 \mid \underline{\omega}_{i,t+1}(k_{i,t+1}), J_t) \\ &= \text{Pr}(\omega_{i,t+1} \geq \underline{\omega}_{i,t+1}(k_{i,t}) \mid \underline{\omega}_{i,t+1}(k_{i,t+1}), \omega_{i,t}) \\ &= \mathscr{P}(\underline{\omega}_{i,t+1}(k_{i,t+1}), \omega_{i,t}) \\ &= \mathscr{P}(i_{i,t}, k_t) \\ &:= P_{i,t} \end{aligned}$$

where $J_{i,t}$ is information set, and where $\mathscr{P}$ approximated non-parametrically.

Olley and Pakes (1996): Details on controlling for selection (2/2) I

Define $g(\underline{\omega}_{i,t+1}(k_{i,t+1}), \omega_{i,t}) := E[\omega_{i,t+1} \mid \omega_{i,t}, \chi_{i,t} = 1]$.

Invert expression $P_{i,t} = \mathscr{P}(\underline{\omega}_{i,t+1}(k_t + 1), \omega_{i,t})$ for survival probability to write g i.t.o. $P_{i,t}$ and $\omega_{i,t}$

Use definition of ϕ in g to transform (46) and estimate

$$y_{i,t+1} - \beta_l l_{i,t+1} = \beta_k k_{i,t+1} + g(P_{i,t}, \phi_{i,t} - \beta_k k_{i,t}) + \xi_{i,t+1} + \eta_{i,t+1}.$$

Finally, use estimates of β_l and β_k to fit (46) and compute productivity as residual.

Appendix

Compustat data used in TFP estimation

I follow İmrohoroğlu and Tuzel (2014) in estimating TFP using Compustat data:

- Compustat fundamental annual data 1962–2015, available through WRDS
- Drop financial firms (SIC 6000–6999) and regulated firms (SIC 4900–4999)
- Deflators: GDP price index, private fixed investment price index, national wage index
- Compustat series: SALE, OIBDP, EMP, PPEGT, DPACT, DP

Calculation of variables:

- Value Added: Sales - Materials
- Materials: Total Expense - Labor Expense
- Total Expense: Sales - Operating Income Before Depreciation and Amortization
- Labor Expense: Number of Employees $\times$ National Average Wage
- Labor Stock: Number of Employees
- Capital Stock: Gross Property, Plant, and Equipment
- Capital Stock Age: Accumulated Depreciation / Current Depreciation, 3-yr average

Appendix

İmrohoroğlu and Tuzel (2014) study the relationship between firm productivity and stock returns

1. Empirical part:
   - Estimate TFP for Compustat firms 1963—2009
   - Descriptive statistics for TFP-sorted portfolios

2. Theoretical part:
   - Partial equilibrium model
   - Productivity heterogeneity
   - Convex adjustment costs

İmrohoroğlu and Tuzel (2014): TFP estimation I

Authors follow a modified version of Olley and Pakes (1996):

$$y_{i,t} = \beta_0 + \beta_k k_{i,t} + \beta_l l_{i,t} + \omega_{i,t} + \eta_{i,t} + controls.$$

where

$\omega_{i,t}$	forecastable by firm, unobserved by econometrician
$\eta_{i,t}$	unforecastable, unobserved

Remarks:

- controls remove effects of industry or aggregate TFP
- firm TFP thus uncorrelated with aggregate TFP *by construction*
- consistent with authors' choice of production function in theory part

Olley-Pakes details

İmrohoroğlu and Tuzel (2014): Beta estimation I

Authors estimate basic CAPM equation:

$$r_{p,t} - r_{f,t} = \alpha_p + \beta_p(r_{m,t} - r_{f,t})$$

where

$r_{p,t}$	value-weighted return on TFP-sorted portfolio p
$r_{f,t}$	risk-free return
$r_{m,t}$	return on market portfolio

Remarks:

- market return is value-weighted return on NYSE, AMEX, NASDAQ stock
- risk-free rate is one-month treasury bill
- portfolios consist of Compustat firms sorted on TFP estimates
- Consumption CAPM more appropriate in my model

İmrohoroğlu and Tuzel (2014): Theory I

Authors use firm-level production functions of the form:

$$Y_{i,t} = A_t Z_{i,t} (K_{i,t})^{\alpha_K} (L_{i,t})^{\alpha_L},$$

where $a_t = \log(A_t)$, $z_{i,t} = \log(Z_{i,t})$, and $cov(a_t, z_{i,t}) = cov(z_{i,t}, z_{j,t}) = 0$.

Note well: $cov(a_t + z_{i,t}, a_t) = var(a_t)\ \forall i$. In other words,

- firm and aggregate TFP covary equally across all firms
- differences in exposure to systematic risk not coming from TFP directly
- instead, convex capital adjustment costs drive risk exposure in model
- this distinguishes their approach (common in literature) from mine
- my empirical task: estimate cov b/w firm and aggregate TFP

Appendix

Profit Comovement

Model derivations

Olley-Pakes Productivity

Compustat Dataset

İmrohoroğlu and Tuzel (2014)

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