Online Appendix

On Aggregate Fluctuations, Systemic Risk, and the Covariance of Firm-Level Activity*

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Abstract

This online appendix provides additional materials to support the main paper, and is divided into a theoretical section and an empirical section. The theoretical section provides derivations for the model, and proves the propositions presented in the paper. The empirical section describes the Olley-Pakes productivity estimation procedure used in the paper.

Keywords: Aggregate Fluctuations, Systemic Risk, Heterogeneity, Productivity, Asset Pricing, Diversification, Endogenous Uncertainty

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Contents

List of Figures

List of Tables

A Theoretical Appendix

This appendix provides additional discussion on several aspects of the paper. It provides a basic mathematical discussion of the model presented in Section 3, proofs of the propositions in Section 4, details of the productivity estimation procedure I use. The discussion includes first-order conditions for the decision problems of the representative household and of consumption goods producers, and derivations of the propositions presented in section 4.

A.1 Optimality conditions

Optimality conditions for the representative household. The household solves its utility maximization problem in two stages. The two-stage budgeting procedure is possible here because the period utility function $u(C_s)$ depends only on the basket C_t, and C_t is homogeneous of degree one (Gorman, 1959). Consider the first-stage problem in (11). Eliminate constraint (10) by substituting for I_t in (9). Use the method of Lagrangian multipliers to rewrite the objective function as

$$\mathcal{L} = \mathrm{E}\left[\sum_{s=t}^{\infty} \beta^{s-t} u(C_s) - \beta^{s-t}\lambda_s \left(C_s + K_{s+1} + \int_{\omega\in\Omega} V_s(\omega) S_{s+1}(\omega)\lambda(d\omega) \right.\right.$$
$$\left.\left. - w_s L - (1 + r_s - \delta)K_s - \int_{\omega\in\Omega} [V_s(\omega) - \Pi_s(\omega)] S_s(\omega)\lambda(d\omega) \right)\right]. \quad (1)$$

To get first-order optimality conditions, equate with zero the first derivatives of $\mathcal{L}$ with respect to choice variables C_s, K_{s+1}, $S_{s+1}(\omega)$, and λ_s for arbitrary period s and firm ω. The household's optimal plans for consumption, capital accumulation, and equity shares, respectively, satisfy the following conditions:

$$\mathrm{E}\Big[u'(C_s)\Big] = \mathrm{E}[\lambda_s], \quad (2)$$
$$\mathrm{E}[\lambda_s] = \beta \mathrm{E}[\lambda_{s+1}(1 + r_{s+1} - \delta)], \quad (3)$$
$$\mathrm{E}[\lambda_s V_s(\omega)] = \beta \mathrm{E}[\lambda_{s+1}(V_{s+1}(\omega) + \Pi_{s+1}(\omega))]. \quad (4)$$

The household's stochastic discount factor also derives from these conditions: set $s = t$ and use (2) and (4) to write firm ω's period-t present value as

$$V_t(\omega) = \mathrm{E}_t\left[\left(\beta\frac{u'(C_{t+1})}{u'(C_t)}\right)\Big(V_{t+1}(\omega) + \Pi_{t+1}(\omega)\Big)\right]. \quad (5)$$

The one-period stochastic discount factor is then the first term in the expectation operator:

$m_{t,t+1} = \beta u'(C_{t+1})/u'(C_t)$. Iterate (5) via $V_{t+1}(\omega)$ to get the multi-period stochastic discount factor. For any period $s \geq t$, write the latter as

$$\begin{aligned} m_{t,s} &= m_{t,t+1} \cdot m_{t+1,t+2} \cdots m_{s-1,s} \\ &= \beta \frac{u'(C_{t+1})}{u'(C_t)} \cdot \beta \frac{u'(C_{t+2})}{u'(C_{t+1})} \cdots \beta \frac{u'(C_s)}{u'(C_{s-1})} = \beta^{s-t} \frac{u'(C_s)}{u'(C_t)}. \end{aligned} \tag{6}$$

Next, solve the household's second-stage problem of allocating consumption across varieties $c_t(v,\omega)$ within the aggregate basket C_t. Let $P_t(v,\omega)$ be the nominal price of variety $c_t(v,\omega)$, and P_t be the nominal price of the consumption basket C_t. The household takes the optimal amount of aggregate consumption C_t as given by the first-stage problem, and takes nominal prices as given, and maximizes its consumption of varieties for each unit of expenditure $1 := P_t C_t$, by solving equation (12). Writing the Lagrangian,

$$\mathcal{L} = \left[\int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} \left[c_t(v,\omega) \right]^{\frac{\theta-1}{\theta}} \lambda(dv d\omega) \right]^{\frac{\theta}{\theta-1}} + \lambda_t \left[1 - \int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} P_t(v,\omega) c_t(v,\omega) \lambda(dv d\omega) \right].$$

Taking the first derivative of the Lagrangian with respect to consumption varieties $c_t(v,\omega), c_t(v',\omega')$, and setting equal to zero,

$$\begin{aligned} C_t^{-1} c_t(v,\omega)^{-\frac{1}{\theta}} &= P_t(v,\omega), \\ C_t^{-1} c_t(v',\omega')^{-\frac{1}{\theta}} &= P_t(v',\omega'), \end{aligned}$$

and the ratio of the two optimality conditions yields,

$$\left(\frac{c_t(v,\omega)}{c_t(v',\omega')} \right)^{-\frac{1}{\theta}} = \frac{P_t(v,\omega)}{P_t(v',\omega')}. \tag{7}$$

Using equation (7) in the expenditure constraint in equation (12),

$$\begin{aligned} 1 &= \int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} P_t(v,\omega) c_t(v,\omega) \lambda(dv d\omega) \\ &= \int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} P_t(v,\omega) \left(\frac{P_t(v',\omega')}{P_t(v,\omega)} \right)^{\theta} c_t(v',\omega') \lambda(dv d\omega) \\ &= P_t(v',\omega')^{\theta} c_t(v',\omega') \int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} P_t(v,\omega)^{1-\theta} \lambda(dv d\omega). \end{aligned}$$

Again using equation (7), notice that the aggregate consumption basket can be written

$$C_t = \left[\int_{\Omega} \int_{\mathcal{V}(\omega)} \Big[c_t(v,\omega)\Big]^{\frac{\theta-1}{\theta}} \lambda(dvd\omega) \right]^{\frac{\theta}{\theta-1}}$$

$$= \left[\int_{\Omega} \int_{\mathcal{V}(\omega)} \left[\left(\frac{P_t(v,\omega)}{P_t(v',\omega')} \right)^{-\theta} c_t(v',\omega') \right]^{\frac{\theta-1}{\theta}} \lambda(dvd\omega) \right]^{\frac{\theta}{\theta-1}}$$

$$= P_t(v',\omega')^{\theta} c_t(v',\omega') \left[\int_{\Omega} \int_{\mathcal{V}(\omega)} P_t(v,\omega)^{1-\theta} \lambda(dvd\omega) \right]^{\frac{\theta}{\theta-1}}.$$

Now recall $1 = P_t C_t$, and define $p_t(v,\omega) := P_t(v,\omega)/P_t$. The above expressions imply the following price index and demand curve:

$$1 = \left[\int_{\Omega} \int_{\mathcal{V}(\omega)} \Big[p_t(v,\omega)\Big]^{1-\theta} \lambda(dvd\omega) \right]^{\frac{1}{1-\theta}}, \tag{8}$$

$$c_t(v,\omega) = \Big[p_t(v,\omega)\Big]^{-\theta} C_t. \tag{9}$$

Optimality conditions for consumption goods producers. Consider firm ω's profit maximization problem (5). Eliminate constraints by using (3) and (9) to substitute for $p_t(v,\omega)$ and $y_t(v,\omega)$ in the firm-vintage profit function (4) that appears in (5). Obtain first-order optimality conditions by equating with zero the first derivatives of $\Pi_t(\omega)$ with respect to choice variables $k_t(v,\omega)$ and $l_t(v,\omega)$ for arbitrary vintage v. Firm ω's optimal choice of capital for production with vintage v satisfies

$$k_t(v,\omega) = (\alpha)\left(\frac{\theta-1}{\theta}\right)(Y_t)^{\frac{1}{\theta}}[y_t(v,\omega)]^{\frac{\theta-1}{\theta}}(r_t)^{-1}. \tag{10}$$

Its optimal choice of labor satisfies

$$l_t(v,\omega) = (1-\alpha)\left(\frac{\theta-1}{\theta}\right)(Y_t)^{\frac{1}{\theta}}[y_t(v,\omega)]^{\frac{\theta-1}{\theta}}(w_t)^{-1}. \tag{11}$$

Notice that the optimal capital-labor ratio depends neither on the individual firm nor on the vintage of technology:

$$\frac{k_t(v,\omega)}{l_t(v,\omega)} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{w_t}{r_t}\right). \tag{12}$$

Optimality conditions for capital goods producers. Now consider the profit maximization problem for the capital goods producer. Take derivatives of gross profit with respect to the factors to obtain first-order conditions:

$$r_t = \alpha Z_t (k_t)^{\alpha-1} (l_t)^{1-\alpha}, \tag{13}$$

$$w_t = (1-\alpha) Z_t (k_t)^{\alpha} (l_t)^{-\alpha}. \tag{14}$$

Notice that the the capital-labor ratio in the capital goods sector is again

$$\frac{k_t}{l_t} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{w_t}{r_t}\right). \tag{15}$$

A.2 Main propositions and proofs

1 *Let $z_t(v)$ now be a random preference multiplier. Replace the stochastic production function in equation* (3) *with equation* (3′) *below, and the non-stochastic preferences in equation* (12) *with equation* (12′) *below:*

$$y_t(v,\omega) = z(\omega)\Big[k_t(v,\omega)\Big]^{\alpha}\Big[l_t(v,\omega)\Big]^{1-\alpha}, \tag{3′}$$

$$C_t = \left[\int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} \Big[z_t(v)c_t(v,\omega)\Big]^{\frac{\theta-1}{\theta}} \lambda(dvd\omega)\right]^{\frac{\theta}{\theta-1}}. \tag{12′}$$

Then the propositions of this section remain true after derivation of the appropriate stochastic household demand curve for individual varieties.

Proof. The proof starts by re-deriving the demand-curve, following appendix A.1 but now under the stochastic preferences in equation (12′). The household solves

$$\begin{aligned} &\max_{\{c_t(v,\omega)\}_{v\in\mathcal{V},\omega\in\Omega}} && \left[\int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} \Big[z_t(v)c_t(v,\omega)\Big]^{\frac{\theta-1}{\theta}} \lambda(dvd\omega)\right]^{\frac{\theta}{\theta-1}} \\ &\text{s.t.} && 1 = \int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} P_t(v,\omega)c_t(v,\omega)\lambda(dvd\omega). \end{aligned} \tag{16}$$

The Lagrangian is

$$\mathcal{L} = \left[\int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} \Big[z_t(v)c_t(v,\omega)\Big]^{\frac{\theta-1}{\theta}} \lambda(dvd\omega)\right]^{\frac{\theta}{\theta-1}} + \lambda_t\left[1 - \int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} P_t(v,\omega)c_t(v,\omega)\lambda(dvd\omega)\right].$$

Taking the first derivative of the Lagrangian with respect to consumption varieties $c_t(v,\omega), c_t(v',\omega')$, and setting equal to zero,

$$C_t^{-1}\Big[z_t(v)c_t(v,\omega)\Big]^{-\frac{1}{\theta}} z_t(v) = P_t(v,\omega),$$
$$C_t^{-1}\Big[z_t(v')c_t(v',\omega')\Big]^{-\frac{1}{\theta}} z_t(v') = P_t(v',\omega'),$$

and the ratio of the two optimality conditions yields,

$$\left(\frac{z_t(v)}{z_t(v')}\right)^{\frac{\theta-1}{\theta}} \left(\frac{c_t(v,\omega)}{c_t(v',\omega')}\right)^{-\frac{1}{\theta}} = \frac{P_t(v,\omega)}{P_t(v',\omega')}. \tag{17}$$

The remaining steps of the derivation are straight-forward and follow the derivation in appendix A.1 closely. □

2 *A productivity aggregate over technologies summarizes all of the technological heterogeneity within an individual firm ω:*

$$Z_t(\omega) = \left[\int\limits_{\mathcal{V}(\omega)} \Big[z(\omega)z_t(v)\Big]^{\theta-1}\lambda(dv)\right]^{\frac{1}{\theta-1}}. \tag{18}$$

A productivity aggregate over firms summarizes all of the firm-specific and technological heterogeneity within the consumption goods sector:

$$Z_t = \left[\int\limits_{\Omega} Z_t(\omega)^{\theta-1}\lambda(d\omega)\right]^{\frac{1}{\theta-1}}. \tag{19}$$

Aggregate factor demands, production, and profit can be written in terms of aggregate productivities and variables that either do not vary across firms, in the case of firm aggregates, or do not vary across firms or technologies, in the case of economy-wide aggregates.

Proof. The household and capital goods producer are representative, so aggregation pertains only to the final goods sector.

Start with the optimality conditions (10) and (11) from the firm's decision problem (5). These expressions contain vintage-specific variables $k_t(v,\omega)$, $l_t(v,\omega)$, and $y_t(v,\omega)$ as well as variables and parameters common to all vintages. Combine equations (10) and (11) with the production function (3) to obtain expressions for $k_t(v,\omega)$, $l_t(v,\omega)$, and $y_t(v,\omega)$

in terms of $z_t(v)$ and variables and parameters common to all vintages:

$$k_t(v,\omega) = [z(\omega)z_t(v)]^{\theta-1}(Y_t)\left(\frac{\theta-1}{\theta}\right)^{\theta}\left(\frac{r_t}{\alpha}\right)^{\alpha(1-\theta)-1}\left(\frac{w_t}{1-\alpha}\right)^{(1-\alpha)(1-\theta)}, \tag{20}$$

$$l_t(v,\omega) = [z(\omega)z_t(v)]^{\theta-1}(Y_t)\left(\frac{\theta-1}{\theta}\right)^{\theta}\left(\frac{r_t}{\alpha}\right)^{\alpha(1-\theta)}\left(\frac{w_t}{1-\alpha}\right)^{(1-\alpha)(1-\theta)-1}, \tag{21}$$

$$y_t(v,\omega) = [z(\omega)z_t(v)]^{\theta}(Y_t)\left(\frac{\theta-1}{\theta}\right)^{\theta}\left(\frac{r_t}{\alpha}\right)^{-\alpha\theta}\left(\frac{w_t}{1-\alpha}\right)^{-(1-\alpha)\theta}. \tag{22}$$

These expressions can be simplified further using an expression derived from the definition of the consumption basket, along with (22) and market clearing:

$$\begin{aligned} Y_t &= \left[\int_{\Omega}\int_{\mathcal{V}(\omega)} [y_t(v,\omega)]^{\frac{\theta-1}{\theta}}\lambda(dvd\omega)\right]^{\frac{\theta}{\theta-1}} \\ &= \left(\frac{\theta-1}{\theta}\right)^{\theta}\left(\frac{\alpha}{r_t}\right)^{\alpha\theta}\left(\frac{1-\alpha}{w_t}\right)^{(1-\alpha)\theta}(Y_t)\left[\int_{\Omega}\int_{\mathcal{V}(\omega)} (z(\omega)z_t(v))^{\theta-1}\lambda(dvd\omega)\right]^{\frac{\theta}{\theta-1}} \end{aligned}$$

$$\Leftrightarrow \qquad Z_t := \left[\int_{\Omega}\int_{\mathcal{V}(\omega)} (z(\omega)z_t(v))^{\theta-1}\lambda(dvd\omega)\right]^{\frac{1}{\theta-1}} = \left(\frac{\theta}{\theta-1}\right)\left(\frac{r_t}{\alpha}\right)^{\alpha}\left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}.$$

Now use the expression for Z_t to simplify (20)–(22):

$$\begin{aligned} k_t(v,\omega) &= \left(\frac{\theta-1}{\theta}\right)\left(\frac{\alpha}{r_t}\right)\left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta-1} Y_t \\ l_t(v,\omega) &= \left(\frac{\theta-1}{\theta}\right)\left(\frac{1-\alpha}{w_t}\right)\left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta-1} Y_t \\ y_t(v,\omega) &= \left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta} Y_t. \end{aligned}$$

Now recall that $p_t(v,\omega) = (y_t(v,\omega)/Y_t)^{-(1/\theta)}$, and use above to get a similar expression for profit:

$$\begin{aligned} \pi_t(v,\omega) &= p_t(v,\omega)y_t(v,\omega) - r_t k_t(v,\omega) - w_t l_t(v,\omega) \\ &= \frac{1}{\theta}\left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta-1} Y_t. \end{aligned}$$

To get firm aggregates, sum the $k_t(v,\omega)$'s, $l_t(v,\omega)$'s, and $\pi_t(v,\omega)$'s, and use the

Dixit-Stiglitz aggregator on $y_t(v,\omega)$:

$$
\begin{aligned}
K_t(\omega) &:= \int\limits_{\mathcal{V}(\omega)} k_t(v,\omega)\lambda(dv) = \left(\frac{\theta-1}{\theta}\right)\left(\frac{\alpha}{r_t}\right)\left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta-1} Y_t \\
L_t(\omega) &:= \int\limits_{\mathcal{V}(\omega)} l_t(v,\omega)\lambda(dv) = \left(\frac{\theta-1}{\theta}\right)\left(\frac{1-\alpha}{w_t}\right)\left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta-1} Y_t \\
Y_t(\omega) &:= \left[\int\limits_{\mathcal{V}(\omega)} (y_t(v,\omega))^{\frac{\theta-1}{\theta}}\lambda(dv)\right]^{\frac{\theta}{\theta-1}} = \left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta} Y_t \\
\Pi_t(\omega) &:= \int\limits_{\mathcal{V}(\omega)} \pi_t(v,\omega)\lambda(dv) = \frac{1}{\theta}\left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta-1} Y_t,
\end{aligned}
$$

where

$$
Z_t(\omega) := \left[\int\limits_{\mathcal{V}(\omega)} (z(\omega)z_t(v))^{\theta-1}\lambda(dv)\right]^{\frac{1}{\theta-1}}.
$$

Further rearrangement along the same lines yields the economy-wide aggregates. It is also possible to write aggregate output in terms of a Cobb-Douglas aggregate production function, at both the firm and economy-wide levels:

$$
Y_t(\omega) = Z_t(\omega)[K_t(\omega)]^{\alpha}[L_t(\omega)]^{1-\alpha} \tag{23}
$$

$$
Y_t = Z_t(K_t)^{\alpha}(L_t)^{1-\alpha}, \tag{24}
$$

where the production function arguments should be understood as *optimal* factor inputs that satisfy the firm's optimality conditions for from the profit maximization problem (see Felipe and Fisher, 2003, for a discussion).

Notice that the firm-level aggregate production function takes the familiar Cobb-Douglas form. But remember that the distribution of shocks is endogenous, and the underlying technology choice problem imposes additional structure on the firm-level productivity multipliers. In particular, if technology sets $\mathcal{V}(\omega)$ differs across firms, so too will the distributions of the random productivity multipliers. And to the extent that technology sets share common elements, firm-level productivity will covary. The next three propositions make these statements rigorous. □

3 *In non-stochastic steady state, any firm ω with productivity $z(\omega) \geq \underline{z}$ chooses technology set $\mathcal{V}(\omega) = \left\{v \in \mathcal{V} : \underline{v} \leq v \leq \overline{v}(\omega)\right\}$, where the endogenous cut-offs $\underline{z}$ and $\overline{v}(\omega)$*

are given by:

$$\underline{z} = \left(\frac{\theta}{\mu_\epsilon}\right)^{\frac{1}{\theta-1}} \tag{25}$$

$$\overline{v}(\omega) = \left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\gamma}} z(\omega)^{\frac{\theta-1}{\gamma}}. \tag{26}$$

Firms with $z(\omega) < \underline{z}$ do not produce. Under parameter restrictions, firms ω_1 and ω_2 with productivities $\underline{z} < z(\omega_1) < z(\omega_2)$ choose technology sets such that $\mathcal{V}_t(\omega_1) \subset \mathcal{V}_t(\omega_2)$. The above cut-offs are also first-order approximate to those that obtain in a stochastic environment.

Proof. Firms choose their technology sets $\mathcal{V}(\omega) \subseteq \mathcal{V} = [\underline{v}, \infty) \subseteq \mathbb{R}^+$ to maximize profit. Recall that technologies differ in their period fixed costs, but not their first two moments. Starting from the technology adoption rule in (7), and rearranging:

$$\begin{aligned} 0 &< \mathrm{E}_t\Big[m_{t,t+1}(\pi_t(v,\omega) - f_s(v))\Big] \\ &= \mathrm{E}_t\left[\beta\frac{u'(C_{t+1})}{u'(C_t)}(\pi_t(v,\omega) - f_s(v))\right] \\ &= \mathrm{E}_t\left[\frac{1}{Y_{t+1}}(\pi_{t+1}(v,\omega) - f_{t+1}(v))\right], \end{aligned}$$

where the third line assumes log utility. Now recall from the proof to 2:

$$\begin{aligned} \pi_t(v,\omega) &= \frac{1}{\theta}\left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta-1} Y_t, \\ f_t(v) &= \frac{Y_t}{\mu}v^\gamma. \end{aligned}$$

Using these expressions in the adoption rule:

$$\begin{aligned} & \mathrm{E}_t\left[\frac{1}{Y_{t+1}}(\pi_{t+1}(v,\omega) - f_{t+1}(v))\right] > 0 \\ \Leftrightarrow \quad & \left(\frac{z(\omega)^{\theta-1}}{\theta}\right)\mathrm{E}_t\left[\left(\frac{z_t(v)}{Z_t}\right)^{\theta-1}\right] \geq \frac{v^\gamma}{\mu}. \end{aligned}$$

From here, either evaluate the productivities in the ratio under the expectation operator at their expected values to get an expression describing steady-state technology sets, or take an approximation of the expression under the expectation operator. A first-order approximation gives the same results as the steady-state solution:

$$\overline{v}(\omega) = \left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\gamma}} z(\omega)^{\frac{\theta-1}{\gamma}}$$

$$\Rightarrow \qquad \underline{z}(v) = \left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\theta-1}} v^{\frac{\gamma}{\theta-1}}.$$

Notice that the cut-off $\overline{v}(\omega)$ increasing in $z(\omega)$, so the more productive firms produce more varieties and use more technology.

Two remarks are in order: First, it is useful that the steady-state and first-order approximate cut-offs coincide, because it means that first-order dynamics around the steady state are completely standard in this model. Second, the second-order approximate case gives more interesting but less tractable results. There is a covariance term in the second-oder approximation that varies with v—covariance is higher for commonly-used technologies. □

4 *Let technology sets be those that firms choose in non-stochastic steady state. Then the first and second moments of firm-level productivity are given by $\mu(\omega)$ and $\sigma^2(\omega)$, respectively:*

$$\mu(\omega) = \mu_\epsilon z(\omega)^{\zeta_{\mu\omega 1}} \left[\left(\frac{z(\omega)}{\underline{z}}\right)^{\zeta_{\mu\omega 2}} - 1\right], \tag{27}$$

$$\sigma^2(\omega) = \sigma_\epsilon^2 z(\omega)^{\zeta_{\sigma\omega 1}} \left[\left(\frac{z(\omega)}{\underline{z}}\right)^{\zeta_{\sigma\omega 2}} - 1\right]. \tag{28}$$

The first and second moments of aggregate productivity are given by μ and σ^2, respectively:

$$\mu = \mu_\epsilon \zeta_{\mu 1} \underline{z}^{\zeta_{\mu 2}}, \tag{29}$$

$$\sigma^2 = \sigma_\epsilon^2 \zeta_{\sigma 1} \underline{z}^{\zeta_{\sigma 2}}. \tag{30}$$

Under parameter restrictions, the first and second moments of all productivity aggregates are positive and finite. For firms ω_1 and ω_2 with $z(\omega_1) < z(\omega_2)$, we have $\mu_t(\omega_1) < \mu_t(\omega_2)$ and $\sigma_t^2(\omega_1) < \sigma_t^2(\omega_2)$.

Proof. Begin with the first moment of sector-aggregate productivity, just using the definition:

$$\mu = \mathrm{E}\Big[Z_t^{\theta-1}\Big] = \mathrm{E}\left[\int\limits_{\Omega} Z_t(\omega)^{\theta-1}\lambda(d\omega)\right]$$

$$= \mathrm{E}\left[\int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} (z(\omega)z_t(v))^{\theta-1}\lambda(dvd\omega)\right]$$

$$= \mathrm{E}\left[\int\limits_{\mathcal{V}}\int\limits_{\Omega_v} (z(\omega)z_t(v))^{\theta-1}\lambda(d\omega dv)\right] = \mathrm{E}\left[\int\limits_{\mathcal{V}} z_t(v)^{\theta-1}\left(\int\limits_{\Omega_v} z(\omega)^{\theta-1}\lambda(d\omega)\right)\lambda(d\omega)\right],$$

where Ω_v is the set of firms using vintage v, that is: $\Omega_v := \{\omega \in \Omega : \underline{z}(v) < z(\omega)\}$, and $\underline{z}(v)$ is the inverse of the cost cut-off $\overline{v}(\omega)$.

Now evaluate the inner integral:

$$\int\limits_{\Omega_v} z(\omega)^{\theta-1}\lambda(d\omega) = \int\limits_{\underline{z}(v)}^{\infty} z(\omega)^{\theta-1}h(z(\omega))dz(\omega)$$

$$= \left[\frac{\kappa}{(\theta-1)-\kappa}z(\omega)^{(\theta-1)-\kappa}\right]_{\underline{z}(v)}^{\infty}$$

$$= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\underline{z}(v)^{(\theta-1)-\kappa}$$

$$= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}}\left(\frac{1}{v}\right)^{\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}.$$

Substitute the evaluated integral back into the expression for μ:

$$\mu = \mathrm{E}\Big[Z_t^{\theta-1}\Big] = \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}}\mathrm{E}\left[\int\limits_{\mathcal{V}} z_t(v)^{\theta-1}v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(dv)\right]$$

$$= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}}\mathrm{E}\left[\int\limits_{\underline{v}}^{\infty} z_t(v)^{\theta-1}v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(dv)\right]$$

Use the definition of technological productivity $z_t(v) := \epsilon_{t,\lceil v\rceil}$, set $\underline{v} = 1$, and write the remaining integral as:

$$
\begin{aligned}
\mathrm{E}\left[\int_{\underline{v}}^{\infty} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv)\right] &= \mathrm{E}\left[\int_{\underline{v}}^{\infty} \epsilon_{t,\lceil v \rceil}^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv)\right] \\
&= \mathrm{E}\left[\int_{1}^{2} \epsilon_{t,2}^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \int_{2}^{3} \epsilon_{t,3}^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \dots\right] \\
&= \mu_\epsilon \int_{1}^{2} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \mu_\epsilon \int_{2}^{3} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \dots .
\end{aligned}
$$

Now consider the integrals of the form:

$$
\begin{aligned}
\int_{n}^{n+1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) &= \left[\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) v^{\frac{-\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right]_{n}^{n+1} \\
&= \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right)\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right].
\end{aligned}
$$

Returning to the expression for μ:

$$
\begin{aligned}
\mu = \mathrm{E}\left[Z_t^{\theta-1}\right] &= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}} \mu_\epsilon \sum_{n=1}^{\infty}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right) \\
&\quad \times \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right] \\
&= \mu_\epsilon \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}}.
\end{aligned}
$$

Notice that $\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\theta-1}}$ appears on the right-hand side. Substituting it for $\underline{z}$, and collecting parameters,

$$
\mu = \mu_\epsilon \zeta_{\mu 1} \underline{z}^{\zeta_{\mu 2}},
$$

where

$$
\begin{aligned}
\zeta_{\mu 1} &:= \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)\left(\frac{\kappa}{\kappa-(\theta-1)}\right) \\
\zeta_{\mu 2} &:= \kappa-(\theta-1)
\end{aligned}
$$

Now turn to the second moment of sector-aggregate productivity. Starting again with the definition:

$$
\begin{aligned}
\sigma^2 = \mathrm{Var}\Big(Z_t^{\theta-1}\Big) &= \mathrm{Var}\left(\int\limits_{\Omega} Z_t(\omega)^{\theta-1}\lambda(d\omega)\right) = \mathrm{Var}\left(\int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} (z(\omega)z_t(v))^{\theta-1}\lambda(dvd\omega)\right) \\
&= \mathrm{Var}\left(\int\limits_{\mathcal{V}} z_t(v)^{\theta-1}\int\limits_{\Omega_v} z(\omega)^{\theta-1}\lambda(d\omega dv)\right) \\
&= \mathrm{Var}\left(\int\limits_{\mathcal{V}} z_t(v)^{\theta-1}\int\limits_{\underline{z}(v)}^{\infty} z(\omega)^{\theta-1}\frac{\kappa}{z(\omega)^{\kappa+1}}\lambda(dz(\omega)dv)\right) \\
&= \mathrm{Var}\left(\int\limits_{\mathcal{V}} z_t(v)^{\theta-1}\frac{\kappa}{(\theta-1)-\kappa}\underline{z}(v)^{-[\kappa-(\theta-1)]}\lambda(d\omega)\right),
\end{aligned}
$$

where from the third to the fourth line I change measure from Lebesgue to Pareto. Continuing, using $\underline{z}(v) = \left(\frac{\theta}{\mu_\epsilon}\right)^{\frac{1}{\theta-1}} v^{\frac{\gamma}{\theta-1}}$,

$$
\begin{aligned}
\sigma^2 = \mathrm{Var}\Big(Z_t^{\theta-1}\Big) &= \mathrm{Var}\left(\int\limits_{\mathcal{V}}\left(\frac{\kappa}{(\theta-1)-\kappa}\right)\left(\frac{\theta}{\mu_\epsilon}\right)^{-\frac{\kappa-(\theta-1)}{\theta-1}} z_t(v)^{\theta-1} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(d\omega)\right) \\
&= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)^2\left(\frac{\theta}{\mu_\epsilon}\right)^{-2\frac{\kappa-(\theta-1)}{\theta-1}} \mathrm{Var}\left(\int\limits_{\mathcal{V}} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}\right).
\end{aligned}
$$

Now consider the integral:

$$
\begin{aligned}
\int\limits_{\mathcal{V}} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(d\omega) &= \int\limits_{\mathcal{V}} \epsilon_{t,\lceil v\rceil} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(d\omega) = \int\limits_{\overline{v}(\omega)}^{\infty} \epsilon_{t,\lceil v\rceil} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(d\omega) \\
&= \epsilon_{t,2}\int\limits_{1}^{2} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(d\omega) + \epsilon_{t,3}\int\limits_{2}^{3} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(d\omega) + \dots \\
&= \sum_{n=1}^{\infty}\frac{\epsilon_{t,n+1}(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)} \\
&\qquad \times\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}}\right].
\end{aligned}
$$

Returning to the expression for σ^2:

$$\sigma^2 = \mathrm{Var}\Big(Z_t^{\theta-1}\Big) = \left(\frac{\kappa}{\kappa-(\theta-1)}\right)^2 \left(\frac{\theta}{\mu_\epsilon}\right)^{-2\frac{\kappa-(\theta-1)}{\theta-1}}$$
$$\times \mathrm{Var}\left(\sum_{n=1}^{\infty} \frac{\epsilon_{t,n+1}(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)} \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}}\right]\right)$$
$$= \sigma_\epsilon^2 \left(\frac{\kappa}{\kappa-(\theta-1)}\right)^2 \left(\frac{\theta}{\mu_\epsilon}\right)^{-2\frac{\kappa-(\theta-1)}{\theta-1}} \left(\frac{(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)^2$$
$$\times \sum_{n=1}^{\infty} \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}}\right]^2.$$

Notice that $\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\theta-1}}$ appears on the right-hand side. Substituting it for $\underline{z}$, and collecting parameters,

$$\sigma^2 = \sigma_\epsilon^2 \zeta_{\sigma 1} \underline{z}^{\zeta_{\sigma 2}},$$

where

$$\zeta_{\sigma 1} := \left(\frac{\kappa}{\kappa-(\theta-1)}\right)^2 \left(\frac{(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)^2 \sum_{n=1}^{\infty} \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}}\right]^2$$
$$\zeta_{\sigma 2} := 2[\kappa-(\theta-1)].$$

□

5 *Let technology sets be those that firms choose in the non-stochastic steady state. Then the covariance between firm and aggregate productivity, denoted by* $\sigma_{\omega\Omega}(\omega) = \mathrm{Cov}\Big(Z_t(\omega)^{\theta-1}, Z_t^{\theta-1}\Big)$, *is given by*

$$\sigma_{\omega\Omega}(\omega) = z(\omega)^{\theta-1} \zeta_{\omega\Omega 1} \left[1 - \left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega 2}}\right] \tag{31}$$

The covariance between firm and aggregate productivity, expressed as a fraction of firm market value, is approximated to a first order by

$$\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)} \approx \frac{1}{Y_t} \left(\frac{\zeta_{\omega\Omega 1}\left[1 - \left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega 2}}\right]}{\zeta_{V1}\left(\frac{z(\omega)}{\underline{z}}\right)^{\zeta_{V2}} + \zeta_{V3}\left(\frac{1}{z(\omega)}\right)^{\zeta_{V4}} - \left(\frac{1}{\underline{z}}\right)^{\zeta_{V4}}}\right). \tag{32}$$

Under parameter restrictions, covariance-over-value falls for all $z(\omega)$ above a threshold. The ratio also falls in the level of aggregate output.

Proof. To start, identify a specific firm ω_1, use the definitions of $Z_t(\omega_1)$ and Z_t in the covariance expression, and the cut-offs $\underline{z}$ and $\overline{v}(\omega)$ for the integral bounds:

$$\sigma_{\omega\Omega}(\omega) = \text{Cov}\Big(Z_t(\omega_1)^{\theta-1}, Z_t^{\theta-1}\Big) = \text{Cov}\left(\int_{\mathcal{V}_t(\omega_1)} [z(\omega_1)z_t(v)]^{\theta-1}\lambda(dv), \int_{\Omega} Z_t(\omega)^{\theta-1}\lambda(d\omega)\right)$$
$$= \text{Cov}\left(\int_{\underline{v}=1}^{\overline{v}(\omega_1)} [z(\omega)z_t(v)]^{\theta-1}\lambda(dv), \int_{\underline{z}}^{\infty}\int_{\underline{v}=1}^{\overline{v}(\omega)} [z(\omega)z_t(v)]^{\theta-1}\lambda(dvd\omega)\right).$$

Now consider the first integral:

$$\int_{\underline{v}=1}^{\overline{v}(\omega_1)} [z(\omega_1)z_t(v)]^{\theta-1}\lambda(dv) = z(\omega_1)^{\theta-1}\int_{\underline{v}=1}^{\overline{v}(\omega_1)} \epsilon_{t,\lceil v\rceil}\lambda(dv)$$
$$= z(\omega_1)^{\theta-1}\left[\int_1^2 \epsilon_{t,2}\lambda(dv) + \int_2^3 \epsilon_{t,3}\lambda(dv) + \cdots + \int_{\overline{v}(\omega)-1}^{\overline{v}(\omega)} \epsilon_{t,\overline{v}(\omega)}\lambda(dv)\right]$$
$$= z(\omega_1)^{\theta-1}\sum_{n=1}^{\overline{v}(\omega)-1}\epsilon_{t,n+1},$$

where I have assumed w.l.g. that $\overline{v}(\omega) \in \mathbb{N}$.

Now consider the second integral:

$$\int_{\underline{z}}^{\infty}\int_{\underline{v}=1}^{\overline{v}(\omega)} [z(\omega)z_t(v)]^{\theta-1}\lambda(dvd\omega) = \int_{\underline{v}=1}^{\infty} z_t(v)^{\theta-1}\left(\int_{\underline{z}(v)}^{\infty} z(\omega)^{\theta-1}\lambda(d\omega)\right)\lambda(dv)$$
$$= \int_{\underline{v}}^{\infty} z_t(v)^{\theta-1}\left(\int_{\underline{z}(v)}^{\infty} z(\omega)^{\theta-1}h(z(\omega))\lambda(dz(\omega))\right)\lambda(dv)$$
$$= \int_{\underline{v}}^{\infty} z_t(v)^{\theta-1}\frac{\kappa}{\kappa-(\theta-1)}\underline{z}(v)^{-[\kappa-(\theta-1)]}\lambda(dv),$$

where line two changes measure from Lebesgue to Pareto. Continuing with the second integral, using $\underline{z}(v) = \left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{1}{\theta-1}} v^{\frac{\gamma}{\theta-1}}$,

$$\int_{\underline{z}}^{\infty} \int_{\underline{v}=1}^{\overline{v}(\omega)} [z(\omega) z_t(v)]^{\theta-1} \lambda(dv d\omega) = \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}} \int_{\underline{v}}^{\infty} z_t(v)^{\theta-1} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv).$$

Now the single integral on the right-hand side:

$$\begin{aligned}\int_{\underline{v}}^{\infty} z_t(v)^{\theta-1} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) &= \int_{\underline{v}=1}^{\infty} \epsilon_{t,\lceil v \rceil}^{\theta-1} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) \\ &= \epsilon_{t,2} \int_{1}^{2} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \epsilon_{t,3} \int_{2}^{3} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \ldots\end{aligned}$$

.

Now consider the integrals of the form:

$$\begin{aligned}\int_{n}^{n+1} v^{\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) &= \left[\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) v^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right]_{n}^{n+1} \\ &= \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right)\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right].\end{aligned}$$

So the single integral becomes:

$$\begin{aligned}\int_{\underline{v}}^{\infty} z_t(v)^{\theta-1} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) &= \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\ &\quad \times \sum_{n=1}^{\infty} \epsilon_{t,n+1} \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right],\end{aligned}$$

and the second integral becomes:

$$\begin{aligned}\int_{\underline{z}}^{\infty} \int_{\underline{v}=1}^{\overline{v}(\omega)} [z(\omega) z_t(v)]^{\theta-1} \lambda(dv d\omega) &= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}} \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\ &\quad \times \sum_{n=1}^{\infty} \epsilon_{t,n+1} \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right].\end{aligned}$$

Now, recall that $\text{Cov}\Big(\epsilon_{t,n}, \epsilon_{t,m}\Big) = 0 \; \forall \; n \neq m$, and write the desired covariance as:

$$
\begin{aligned}
\sigma_{_{\omega\Omega}}(\omega) &= \text{Cov}\Big(Z_t(\omega_1)^{\theta-1}, Z_t^{\theta-1}\Big) \\
&= z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\
&\times \text{Cov}\left(\sum_{n=1}^{\overline{v}(\omega)-1}\epsilon_{t,n+1}, \sum_{n=1}^{\infty}\epsilon_{t,n+1}\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right]\right) \\
&= z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\
&\quad \times \sum_{n=1}^{\overline{v}(\omega)-1}\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right]\text{Cov}\Big(\epsilon_{t,n+1}, \epsilon_{t,n+1}\Big)
\end{aligned}
$$

,

where $\text{Cov}\Big(\epsilon_{t,n+1}, \epsilon_{t,n+1}\Big) = \sigma_\epsilon^2$.

Notice that the right-hand side summation, with a as a temporary placeholder, is of form:

$$
\begin{aligned}
\sum_{n=1}^{\overline{v}(\omega_1)-1}\left[\left(\frac{1}{n}\right)^a - \left(\frac{1}{n+1}\right)^a\right] &= \left[\left(\frac{1}{1}\right)^a - \left(\frac{1}{2}\right)^a + \left(\frac{1}{2}\right)^a - \left(\frac{1}{3}\right)^a + \cdots - \left(\frac{1}{\overline{v}(\omega_1)}\right)^a\right] \\
&= \left[1 - \left(\frac{1}{\overline{v}(\omega_1)}\right)^a\right].
\end{aligned}
$$

Returning to the covariance expression, and simplifying the summation as above,

$$
\begin{aligned}
\sigma_{_{\omega\Omega}}(\omega) &= \text{Cov}\Big(Z_t(\omega_1)^{\theta-1}, Z_t^{\theta-1}\Big) \\
&= \sigma_\epsilon^2 z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\
&\quad \times \left[1 - \left(\frac{1}{\overline{v}(\omega_1)}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right] \\
&= \sigma_\epsilon^2 z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\
&\quad \times \left[1 - \left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{\gamma[\kappa-(\theta-1)]}{\gamma(\theta-1)}}\left(\frac{1}{z(\omega_1)}\right)^{\frac{\gamma[\kappa-(\theta-1)]}{\gamma}}\right]
\end{aligned}
$$

where the last line uses $\overline{v}(\omega_1) = \left(\frac{\mu_\epsilon}{\theta\mu}\right)^{\frac{1}{\gamma}} z(\omega_1)^{\frac{\theta-1}{\gamma}}$.

Finally, collect parameters, return from specific ω_1 to arbitrary ω, and write:

$$\frac{\sigma_{\omega\Omega}(\omega)}{\sigma_\epsilon^2} = z(\omega)^{\theta-1}\zeta_{\omega\Omega1}\left[1 - \left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega2}}\right], \text{ where}$$

$$\zeta_{\omega\Omega1} = \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1)]}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right)$$

$$\zeta_{\omega\Omega1} = \frac{\gamma[\kappa-(\theta-1)]}{\gamma}.$$

Recall that the μ appearing in $\zeta_{\omega\Omega1}$ has already been expressed in terms of parameters, so the above expression suffices.

Now turn to covariance over market value. Start from the following primitives:

$$V_t(\omega) = \mathrm{E}_t\left[\sum_{s=t+1}^{\infty}\beta^{s-t}\frac{u'(C_s)}{u'(C_t)}\Big(\Pi_s(\omega) - F_s(\omega)\Big)\right]$$

$$\Pi_s(\omega) = \int_{\mathcal{V}(\omega)} \pi_s(v,\omega)\lambda(dv) = \frac{1}{\theta}\left(\frac{Z_s(\omega)}{Z_s}\right)^{\theta-1} Y_s$$

$$F_s(\omega) = \int_{\mathcal{V}(\omega)} \frac{Y_s}{\mu} v^\gamma \lambda(dv).$$

Using $u(C_s) = \ln(C_s)$ and above primitives, rearrange to get:

$$\frac{V_t(\omega)}{Y_t} = \mathrm{E}\left[\sum_{s=t+1}^{\infty}\beta^{s-t}\left\{\frac{1}{\theta}\left(\frac{z(\omega)}{Z_s}\right)^{\theta-1}\int_{\mathcal{V}(\omega)} z_s(v)^{\theta-1} - \frac{v^\gamma}{\mu}\lambda(dv)\right\}\right].$$

Split up the integral and evaluate the first term, assuming w.l.g. that $\overline{v}(\omega) \in \mathbb{N}$:

$$\begin{aligned}
\int_{\mathcal{V}(\omega)} z_s(v)^{\theta-1}\lambda(dv) &= \int_{\underline{v}=1}^{\overline{v}(\omega)} \epsilon_{s,\lceil v\rceil}\lambda(dv) \\
&= \int_1^2 \epsilon_{s,2}\lambda(dv) + \int_2^3 \epsilon_{s,3}\lambda(dv) + \cdots + \int_{\overline{v}(\omega)-1}^{\overline{v}(\omega)} \epsilon_{s,\overline{v}(\omega)}\lambda(dv) \\
&= \sum_{n=1}^{\overline{v}(\omega)-1} \epsilon_{s,n+1}
\end{aligned}$$

Now evaluate the second part of the integral that we split above:

$$\int\limits_{\mathcal{V}(\omega)} \frac{v^{\gamma}}{\mu}\lambda(dv) = \frac{1}{\mu}\left(\frac{\overline{v}(\omega)^{\gamma+1}}{\gamma+1} - \frac{1}{\gamma+1}\right).$$

Substituting back into the expression for firm value,

$$\frac{V_t(\omega)}{Y_t} = \sum_{s=t+1}^{\infty} \beta^{s-t} \frac{z(\omega)^{\theta-1}}{\theta} \sum_{n=1}^{\overline{v}(\omega)-1} \mathrm{E}\left[\frac{\epsilon_{s,n+1}}{Z_s^{\theta-1}}\right] - \sum_{s=t+1}^{\infty} \beta^{s-t}\left(\frac{\overline{v}(\omega)^{\gamma+1}}{1+\gamma} - \frac{1}{1+\gamma}\right)$$

To a first-order approximation, the expectation is: $\mathrm{E}\left[\frac{\epsilon_{s,n+1}}{Z_s^{\theta-1}}\right] \approx \frac{\mu_\epsilon}{\mu}$. Simplifying,

$$\frac{V_t(\omega)}{Y_t} \approx z(\omega)^{\frac{1+\gamma}{\gamma}(\theta-1)}\left(\frac{\mu_\epsilon}{\theta\mu}\right)^{\frac{1+\gamma}{\gamma}}\left(\frac{\gamma}{1+\gamma}\right) - z(\omega)^{\theta-1}\left(\frac{\mu_\epsilon}{\theta\mu}\right) + \left(\frac{1}{1+\gamma}\right)$$

Now combining with the covariance expression derived above:

$$\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)} \approx \frac{\sigma_\epsilon^2}{Y_t} \cdot \frac{z(\omega)^{\theta-1}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)\left[1 - \left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\gamma(\theta-1)}}\left(\frac{1}{z(\omega)}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\gamma}}\right]}{z(\omega)^{\frac{1+\gamma}{\gamma}(\theta-1)}\left(\frac{\mu_\epsilon}{\theta\mu}\right)^{\frac{1+\gamma}{\gamma}}\left(\frac{\gamma}{1+\gamma}\right) - z(\omega)^{\theta-1}\left(\frac{\mu_\epsilon}{\theta\mu}\right) + \frac{1}{1+\gamma}}.$$

Finally, using the expression for $\underline{z}$, and collecting parameters to simplify,

$$\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)} \approx \frac{1}{Y_t}\left(\frac{\zeta_{\omega\Omega 1}\left[1 - \left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega 2}}\right]}{\zeta_{V1}\left(\frac{z(\omega)}{\underline{z}}\right)^{\zeta_{V2}} + \zeta_{V3}\left(\frac{1}{z(\omega)}\right)^{\zeta_{V4}} - \left(\frac{1}{\underline{z}}\right)^{\zeta_{V4}}}\right), \text{ where}$$

$$\zeta_{\omega\Omega 1} := \underline{z}^{\theta-1}\left(\frac{\sigma_\epsilon^2(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right), \qquad \zeta_{\omega\Omega 2} := \left(\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\gamma}\right)$$

$$\zeta_{V1} = \left(\frac{\gamma}{1+\gamma}\right), \qquad \zeta_{V2} := \left(\frac{\theta-1}{\gamma}\right), \qquad \zeta_{V3} := \left(\frac{1}{\gamma+1}\right), \qquad \zeta_{V4} := (\theta-1).$$

□

6 *Let technology sets be those that firms choose in the non-stochastic steady state, and let $Z_t(\omega_1)$ and $Z_t(\omega_2)$ be firm productivities for firms ω_1 and ω_2, where $Z_t(\omega_1) < Z_t(\omega_2)$. Then the correlation between firm productivities is given by*

$$\mathrm{Corr}\left(Z_t(\omega_1), Z_t(\omega_2)\right) = blah, \tag{33}$$

and the correlation $\text{Corr}\left(Z_t(\omega_1), Z_t(\omega_2)\right)$ *is decreasing in the distance between productivities,* $|z(\omega_1) - z(\omega_2)|$.

Proof. □

7 *Let technology sets be those that firms choose in the non-stochastic steady state. Then firm* ω*'s expected excess return is approximated to a second order by*

$$\mathrm{E}_t\Big[r_{t+1}(\omega) - r_{f,t+1}\Big] \approx \zeta_{r1}\frac{\mu(\omega)}{V_t(\omega)} + \zeta_{r2}\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)}, \tag{34}$$

where I define firm ω*'s return as* $r_t(\omega) = \Big[V_{t+1}(\omega) + \Pi_{t+1}(\omega) - F_{t+1}(\omega)\Big]/V_t(\omega)$*, and the risk-free rate as* $r_{f,t} = m_{t,t+1}^{-1}$*. Under parameter restrictions, expected excess returns decrease in firm productivity* $z(\omega)$ *for all* $z(\omega)$ *above a threshold.*

Proof. Start with the definition of firm ω's stock return:

$$\begin{aligned}
r_{t+1}(\omega) &= \frac{V_{t+1}(\omega) + \Pi_{t+1}(\omega) - F_{t+1}(\omega)}{V_t(\omega)} \\
&= \frac{\mathrm{E}\Big[\sum_{s=t+2}^{\infty} m_{t+1,s}(\Pi_s(\omega) - F_s(\omega))\Big] + \Pi_{t+1}(\omega) - F_{t+1}(\omega)}{V_t(\omega)} \\
&= \frac{Y_{t+1}\mathrm{E}\Big[\sum_{s=t+1}^{\infty} \beta^{s-(t+1)}\Big(\frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s}\Big)\Big]}{V_t(\omega)},
\end{aligned}$$

where the third line assumes log utility and uses the definition of the household stochastic discount factor. Now take the time-t conditional expectation:

$$\begin{aligned}
\mathrm{E}_t\left[r_{t+1}(\omega)\right] &= \mathrm{E}_t\left[\frac{Y_{t+1}\mathrm{E}_{t+1}\Big[\sum_{s=t+1}^{\infty} \beta^{s-(t+1)}\Big(\frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s}\Big)\Big]}{V_t(\omega)}\right] \\
&= \frac{\mathrm{E}_t[Y_{t+1}]\mathrm{E}_t\Big[\mathrm{E}_{t+1}\Big[\sum_{s=t+1}^{\infty} \beta^{s-(t+1)}\Big(\frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s}\Big)\Big]\Big]}{V_t(\omega)} \\
&\qquad + \frac{\mathrm{Cov}_t\Big(Y_{t+1}, \mathrm{E}_{t+1}\Big[\sum_{s=t+1}^{\infty} \beta^{s-(t+1)}\Big(\frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s}\Big)\Big]\Big)}{V_t(\omega)} \\
&= \frac{\mathrm{E}_t[Y_{t+1}]}{\beta Y_t} + \frac{\mathrm{Cov}_t\Big(Y_{t+1}, \mathrm{E}_{t+1}\Big[\sum_{s=t+1}^{\infty} \beta^{s-(t+1)}\Big(\frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s}\Big)\Big]\Big)}{V_t(\omega)} \\
\Leftrightarrow \quad \mathrm{E}_t\Big[r_{t+1}(\omega) - r_{f,t+1}\Big] &= \frac{\mathrm{Cov}_t\Big(Y_{t+1}, \mathrm{E}_{t+1}\Big[\sum_{s=t+1}^{\infty} \beta^{s-(t+1)}\Big(\frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s}\Big)\Big]\Big)}{V_t(\omega)}.
\end{aligned}$$

Consider the covariance term separately, recalling that zero serial correlation is assumed for $\epsilon_{s,n}$s:

$$\text{Cov}_t\left(Y_{t+1}, \text{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s}\right)\right]\right) = \text{Cov}_t\left(Y_{t+1}, \left(\frac{\Pi_{t+1}(\omega)}{Y_{t+1}} - \frac{F_{t+1}(\omega)}{Y_{t+1}}\right)\right)$$
$$= \text{E}_t[\Pi_{t+1}(\omega) - F_{t+1}(\omega)] - \text{E}_t[Y_{t+1}]\text{E}_t\left[\frac{\Pi_{t+1}(\omega)}{Y_{t+1}} - \frac{F_{t+1}(\omega)}{Y_{t+1}}\right]$$
$$= \text{E}_t\left[Y_{t+1}\int_{\mathcal{V}(\omega)}\frac{1}{\theta}\left(\frac{z(\omega)z_{t+1}(v)}{Z_{t+1}}\right)^{\theta-1}\lambda(dv)\right] - \text{E}_t[Y_{t+1}]\text{E}_t\left[\int_{\mathcal{V}(\omega)}\frac{1}{\theta}\left(\frac{z(\omega)z_{t+1}(v)}{Z_{t+1}}\right)^{\theta-1}\lambda(dv)\right]$$
$$= \text{E}_t\left[\frac{1}{\theta}Y_{t+1}\left(\frac{Z_{t+1}(\omega)}{Z_{t+1}}\right)^{\theta-1}\right] - \text{E}_t[Y_{t+1}]\text{E}_t\left[\frac{1}{\theta}\left(\frac{Z_{t+1}(\omega)}{Z_{t+1}}\right)^{\theta-1}\right],$$

where the second line uses the assumption of zero serial correlation. Now recall from (23) that $Y_{t+1} = Z_{t+1}(K_{t+1})^{\alpha}(L)^{1-\alpha}$, that K_{t+1} is determined in t, and that L fixed, so

$$\text{Cov}_t\left(Y_{t+1}, \text{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s}\right)\right]\right)$$
$$= \frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left(\text{E}_t\left[\frac{Z_{t+1}(\omega)^{\theta-1}}{Z_{t+1}^{\theta-2}}\right] - \text{E}_t[Z_{t+1}]\text{E}_t\left[\left(\frac{Z_{t+1}(\omega)}{Z_{t+1}}\right)^{\theta-1}\right]\right)$$

Next, second-order approximate the individual right-hand side expectations around the non-stochastic steady state values $\mu(\omega)$ and μ. Starting with the first right-hand side expectation:

$$\text{E}_t\left[\frac{Z_{t+1}(\omega)^{\theta-1}}{Z_{t+1}^{\theta-2}}\right] \approx \frac{\mu(\omega)}{\mu^{\frac{\theta-2}{\theta-1}}} + \frac{1}{2}\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{\theta-2}{\theta-1}+1\right)\frac{\mu(\omega)}{\mu^{\frac{\theta-2}{\theta-1}+2}}\cdot\sigma^2 - \frac{1}{2}\left(\frac{\theta-2}{\theta-1}\right)\frac{1}{\mu^{\frac{\theta-2}{\theta-1}+1}}\cdot\sigma_{\omega\Omega}(\omega).$$

Now the second:

$$\text{E}_t[Z_{t+1}] \approx \mu^{\frac{1}{\theta-1}} + \frac{1}{2}\left(\frac{1}{\theta-1}\right)\left(\frac{1}{\theta-1}-1\right)\mu^{\frac{1}{\theta-1}-2}\sigma^2.$$

And the third:

$$\text{E}_t\left[\left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta-1}\right] \approx \left(\frac{1}{\mu}+\frac{\sigma^2}{\mu^3}\right)\mu(\omega) - \left(\frac{1}{2\mu^2}\right)\sigma_{\omega\Omega}(\omega).$$

Substituting the approximations in the covariance expression, and rearranging,

$$
\begin{aligned}
&\mathrm{Cov}_t\left(Y_{t+1}, \mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right)\\
&\approx \frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left(\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{\mu^{\frac{1}{\theta-1}}}{\mu}\right)\sigma^2+\left(\frac{\mu^{\frac{1}{\theta-1}}}{\mu}\right)\left(\frac{\sigma}{\mu}\right)^2\left[\left(\frac{1}{2}\right)\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{1}{\theta-1}\right)\left(\frac{\sigma}{\mu}\right)^2-1\right]\right)\cdot\mu(\omega)\\
&\qquad+\frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left[\left(\frac{1}{2}\right)\left(\frac{1}{\theta-1}\right)\left(\frac{1}{\mu^{\frac{1}{\theta-1}}}\right)\left\{1-\left(\frac{1}{2}\right)\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{1}{\mu^2}\right)\right\}\right]\cdot\sigma_{\omega\Omega}(\omega)\\
&\approx \zeta_{r1}\mu(\omega)+\zeta_{r2}\sigma_{\omega\Omega}(\omega),
\end{aligned}
$$

where ζ_{r1} and ζ_{r2} are parameter collections given by:

$$
\begin{aligned}
\zeta_{r1} &:= \frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left(\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{\mu^{\frac{1}{\theta-1}}}{\mu}\right)\sigma^2+\left(\frac{\mu^{\frac{1}{\theta-1}}}{\mu}\right)\left(\frac{\sigma}{\mu}\right)^2\left[\left(\frac{1}{2}\right)\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{1}{\theta-1}\right)\left(\frac{\sigma}{\mu}\right)^2-1\right]\right)\\
\zeta_{r2} &:= \frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left[\left(\frac{1}{2}\right)\left(\frac{1}{\theta-1}\right)\left(\frac{1}{\mu^{\frac{1}{\theta-1}}}\right)\left\{1-\left(\frac{1}{2}\right)\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{1}{\mu^2}\right)\right\}\right],
\end{aligned}
$$

and K_{t+1} is evaluated at its steady-state value. Finally, returning to the expression for expected excess returns,

$$
\mathrm{E}_t\left[r_{t+1}(\omega)-r_{f,t+1}\right]\approx\zeta_{r1}\frac{\mu(\omega)}{V_t(\omega)}+\zeta_{r2}\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)}.
$$

□

A.3 Steady-state equilibrium

Equilibrium requires that the following market clearing conditions hold:

$$
\begin{aligned}
c_t(v,\omega) &= y_t(v,\omega),\\
L &= \int_{\Omega}\int_{\mathcal{V}(\omega)} l_t(v,\omega)\lambda(dvd\omega)+l_t,\\
K_t &= \int_{\Omega}\int_{\mathcal{V}(\omega)} k_t(v,\omega)\lambda(dvd\omega)+k_t,\\
\tilde{I}_t &= I_t+\int_{\Omega}\int_{\mathcal{V}(\omega)} f_s(v)\lambda(dvd\omega),\\
S_t(\omega) &= 1.
\end{aligned}
$$

In the steady state equilibrium, random productivities take their expected values

$(z_t(v)^{\theta-1} = \mu_\epsilon,\ \forall v \in \mathcal{V})$, and capital and consumption are constant over time ($C_{t+1} = C_t = C^*, K_{t+1} = K_t = K^*$). Under these conditions, solving for steady-state values of endogenous variables is straight forward.

Begin by solving for the steady state wage and rental rate. In steady state, (3) becomes $1 = \beta(1 + r^* - \delta)$. Using (13) to substitute for r^*:

$$\begin{aligned} 1 &= \beta(1 + r^* - \delta) \\ &= \beta(1 - \delta + \alpha\mu\left(\frac{l^*}{k^*}\right)^{1-\alpha}) \\ \Leftrightarrow \qquad \frac{k^*}{l^*} &= \left[\frac{\alpha\beta\mu}{1 - \beta(1-\delta)}\right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

Returning to (13) and evaluating at steady state,

$$\begin{aligned} r^* &= \alpha\mu\left(\frac{k^*}{l^*}\right)^{\alpha-1} \\ &= \mu\left[\frac{\alpha\beta\mu}{1 - \beta(1-\delta)}\right]^{\frac{\alpha-1}{1-\alpha}} \\ &= \frac{1 - \beta(1-\delta)}{\beta}. \end{aligned}$$

Now using (14),

$$\begin{aligned} w^* &= (1-\alpha)\mu\left(\frac{k^*}{l^*}\right)^{\alpha} \\ &= (1-\alpha)\mu\left[\frac{\alpha\beta\mu}{1 - \beta(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}}. \end{aligned}$$

Combining,

$$\frac{r^*}{w^*} = \left(\frac{\alpha}{1-\alpha}\right)\left[\frac{1 - \beta(1-\delta)}{\alpha\beta\mu}\right]^{\frac{1}{1-\alpha}}.$$

Next, find an expression for the steady-state aggregate capital stock. Start with the definitions of aggregate capital and labor:

$$\begin{aligned} K^* &= \int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} k^*(v,\omega)\lambda(dvd\omega) + k^*, \\ L &= \int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} l^*(v,\omega)\lambda(dvd\omega) + l^*. \end{aligned}$$

Now use (12) and (15) to write

$$L = \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{w^*}{r^*}\right)\left[\int_{\Omega}\int_{\mathcal{V}(\omega)} k^*(v,\omega)\lambda(dvd\omega) + k^*\right]$$

$$\Leftrightarrow \qquad K^* = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{r^*}{w^*}\right)L$$

$$= \left(\frac{\alpha}{1-\alpha}\right)^2\left[\frac{1-\beta(1-\delta)}{\alpha\beta\mu}\right]^{\frac{1}{1-\alpha}} L.$$

Recall that L is exogenous, so the above expression suffices. Next, use the law of motion for capital to find a steady-state expression for investment demand I_t:

$$K^* = I^* + (1-\delta)K^*$$

$$\Leftrightarrow I^* = \delta K^*$$

$$= \delta\left(\frac{\alpha}{1-\alpha}\right)^2\left[\frac{1-\beta(1-\delta)}{\alpha\beta\mu}\right]^{\frac{1}{1-\alpha}} L.$$

B Empirical Appendix

I follow the procedures in Olley and Pakes (1996); İmrohoroğlu and Tuzel (2014), and estimate a Cobb-Douglas production function in log form. The estimation equation is given by:

$$\ln(Y_{\omega,t}) = \alpha_0 + \alpha_K \ln(K_{\omega,t}) + \alpha_L \ln(L_{\omega,t}) + \ln(Z_{\omega,t}), \tag{35}$$

where the residual $Z_{\omega,t}$ is firm-level total factor productivity. The procedure assumes $Z_{\omega,t} = \xi_{\omega,t}\eta_{\omega,t}$, where $Z_{\omega,t}$ is unknown to the econometrician, but $\xi_{\omega,t}$ is known to the firm.

Olley and Pakes (1996) use a simple behavioral model to derive reduced-form decision rules for firms deciding each period whether to exit or continue producing, and if continuing, how much new capital to purchase. Firms' decision rules depend on their current knowledge of productivity $\xi_{\omega,t}$. Each firm's exit decision is captured by an indicator function $\chi_{\omega,t}$:

$$\chi_{\omega,t} = \begin{cases} 1 & \text{if } \xi_{\omega,t} > \underline{\xi}_{\omega,t} \\ 0 & \text{otherwise,} \end{cases} \tag{36}$$

and their investment decision is captured by an investment function:

$$\ln(I_{\omega,t}) = \ln(I_{\omega,t})\Big(\ln(\xi_{\omega,t}), \ln(K_{\omega,t})\Big). \tag{37}$$

Inverting the investment function, $\ln(\xi_{\omega,t}) = \ln(\xi_{\omega,t})\Big(\ln(K_{\omega,t}), \ln(I_{\omega,t})\Big)$. Defining a new function, $\phi\Big(\ln(K_{\omega,t}), \ln(I_{\omega,t})\Big) = \alpha_0 + \alpha_K \ln(K_{\omega,t}) + \ln(\xi_{\omega,t})\Big(\ln(K_{\omega,t}), \ln(I_{\omega,t})\Big)$, the productivity regression equation (35) becomes

$$\ln(Y_{\omega,t}) = \alpha_L \ln(L_{\omega,t}) + \phi\Big(\ln(K_{\omega,t}), \ln(I_{\omega,t})\Big) + \ln(\eta_{\omega,t}), \tag{38}$$

In a first stage regression, equation (38) is estimated by least squares, where the function $\phi\Big(\ln(K_{\omega,t}), \ln(I_{\omega,t})\Big)$ controls for the forecastable component of firm productivity, and is approximated by a polynomial in $\ln(K_{\omega,t})$ and $\ln(I_{\omega,t})$, denoting a firm's capital stock and investment. I include time-industry controls in this stage to prevent time-industry effects from influencing the first-stage estimates. The remaining estimation equations are given by:

$$P_{i,t} = \mathscr{P}_t(i_{i,t}, k_{i,t}) \tag{39}$$

$$\ln(Y_{\omega,t}) - \alpha_L \ln(L_{\omega,t}) = \alpha_L \ln(L_{\omega,t}) + g(P_{i,t}, \phi_{i,t} - \beta_k k_{i,t}) + \ln(\xi_{\omega,t+1}) + \ln(\eta_{\omega,t+1}). \tag{40}$$

In the second stage, each firm's probability of exit is estimated by equation (39) using probit, where $\mathscr{P}_t(i_{i,t}, k_{i,t})$ is approximated by a polynomial in i_t and k_t. Finally, equation (40) is estimated by non-linear least squares, using estimates from stages one and two for $P_{i,t}$ and $\phi_{i,t}$, and approximating $g(P_{i,t}, \phi_{i,t} - \beta_k k_{i,t})$ non-parametrically. The non-parametric functions ϕ, $\mathscr{P}$, and g are derived in greater detail in Olley and Pakes (1996).

I map model variables to Compustat variables in the following way, writing Compustat variables in fixed-width font: labor expense is $L_{\omega,t} = \texttt{WAGE} \times \texttt{EMP}$; capital is $K_{\omega,t} = \texttt{L.PPENT}$, value added is $Y_{\omega,t} = \texttt{OIBDP} + \texttt{WAGE} \times \texttt{EMP}$.

İmrohoroğlu and Tuzel (2014) use an expanding estimation window to prevent information that would have been unavailable to market participants in a particular period from distorting results when they combine estimated productivity with financial market data. I find that the expanding window approach leads to large differences in the volatility of production function estimates in earlier periods relative to later periods. This increased volatility biases the rolling-window covariance estimates in early years, so I instead use the full sample period to estimate production function parameters, and then compute productivity as the residual each period.

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